Subsequences and the Bolzano-Weierstrass Theorem
• **A subsequence** of a sequence \((x_n)_{n \in \mathbb{N}}\) is a particular sequence whose terms are selected among those of the mother sequence \((x_n)_{n \in \mathbb{N}}\).

• The study of subsequences is important because it provides vital information about the convergence of the parent sequence.

• For example, we will see that a sequence which contains two subsequences converging to different limits is divergent.

• An interesting result in this section (**Bolzano-Weierstrass Theorem**) states that every bounded sequence has a convergent subsequence.

• **Plan**

  – Subsequences and properties
  – The Bolzano-Weierstrass Theorem
Definition 1  Given a sequence \((x_n)_{n \in \mathbb{N}}\) of real numbers, let

\[ n_1 < n_2 < \cdots < n_k < \cdots \]

be a strictly increasing sequence of natural numbers. Then the sequence given by

\[
(x_{n_1}, x_{n_2}, \cdots, x_{n_k}, \cdots) = (x_{n_k})_{k \in \mathbb{N}}
\]

is called a subsequence of \((x_n)_{n \in \mathbb{N}}\).
Example 2

- Let

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & 2 & \cdots & k \\
\end{pmatrix}
\]

be a sequence. Then

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
2 & 1 & 2 & \cdots & 2 & k \\
\end{pmatrix}
\]

is a subsequence of the given sequence.

- Let

\[
(2^1, 2^2, \cdots, 2^k, \cdots)
\]

be a sequence. Then

\[
(2^{2^1}, 2^{2^2}, \cdots, 2^{2^k}, \cdots)
\]

is a subsequence of the given sequence.
• Let \((x_n)_{n \in \mathbb{N}}\) be a sequence such that

\[ x_n = (-1)^n \frac{1}{n}. \]

Then \((x_{2k})_{k \in \mathbb{N}}\) is a subsequence given by

\[ x_{2k} = (-1)^{2k} \frac{1}{2k} = \frac{1}{2k}. \]

Also, \((x_{2k+1})_{k \in \mathbb{N}}\) is a subsequence given by

\[ x_{2k+1} = (-1)^{2k+1} \frac{1}{2k + 1} = \frac{1}{2k + 1}. \]
Let \((1 \ 1 \ \ldots \ 1)\) be a sequence. Then
\[
\left(\frac{1}{\cos(1)}, \frac{1}{\cos(2)}, \ldots, \frac{1}{\cos(k)}, \ldots\right)
\]
is a not subsequence of the given sequence because
\[(\cos(k))_{k \in \mathbb{N}}\]
is not an increasing function.
**Test your understanding:** You should pause here, and (1) find an example of a sequence (2) write down an example of a subsequence of the sequence given in (1) and (2) find an example of a sequence which is not a subsequence of the sequence given in (1)

**Theorem 3** If \((x_n)_{n \in \mathbb{N}}\) is a sequence of real numbers which is convergent to \(x\) then any subsequence \((x_{n_k})_{k \in \mathbb{N}}\) of \((x_n)_{n \in \mathbb{N}}\) is convergent to \(x\).
Proof. Let $\epsilon > 0$. By assumption, there exists a natural number $N$ depending on $\epsilon$ such that if $n > N$ then

$$|x_n - x| < \epsilon.$$ 

I claim that since $(n_k)_{k \in \mathbb{N}}$ is an increasing sequence then it is true that $n_k \geq k$ for every natural number $k$.

Indeed by induction we known that $n_1 \geq 1$ since $n_1$ is a natural number. Next, suppose that $n_\ell \geq \ell$ for some $\ell \in \mathbb{N}, \ell \geq 1$.

Since $(n_k)_{k \in \mathbb{N}}$ is increasing then

$$n_{\ell+1} > n_\ell \quad \text{(strict inequality is important here!!!)}$$ 

Now by the inductive hypothesis,

$$n_\ell \geq \ell$$ 

and it follows that $n_{\ell+1} > \ell$.

Since $n_{\ell+1}, \ell$ are natural numbers

$$n_{\ell+1} \geq \ell + 1.$$ 

Moving forward, if $k > N$ then

$$n_k \geq k > N$$ 

and consequently,

$$|x_{n_k} - x| < \epsilon$$ 

Thus,

$$\lim_{k \to \infty} x_{n_k} = x$$

\[ \blacksquare \]
Corollary 4 If \((x_n)_{n \in \mathbb{N}}\) is a sequence of real numbers admitting two subsequences which converge to different limits, then \((x_n)_{n \in \mathbb{N}}\) is divergent.
Example 5 Let us consider the following sequence
\[ a_n = (-1)^n \left( 2 + \frac{1}{n} \right). \]

Then
\[ a_{2k} = (-1)^{2k} \left( 2 + \frac{1}{2k} \right) = 2 + \frac{1}{2k} \]
and
\[ a_{2k+1} = (-1)^{2k+1} \left( 2 + \frac{1}{2k+1} \right) = - \left( 2 + \frac{1}{2k+1} \right) \]

Now note that
\[ \lim_{k \to \infty} a_{2k} = \lim_{k \to \infty} 2 + \frac{1}{2k} = \lim_{k \to \infty} 2 + \lim_{k \to \infty} \frac{1}{2k} = 2 \]
\[ \lim_{k \to \infty} a_{2k+1} = \lim_{k \to \infty} -2 - \frac{1}{2k+1} = \lim_{k \to \infty} (-2) + \lim_{k \to \infty} \left( -\frac{1}{2k+1} \right) = -2 \]

Thus, \((a_n)_{n \in \mathbb{N}}\) has two subsequences converging to two different limits. As such \((a_n)_{n \in \mathbb{N}}\) is a divergent sequence.
Example 6 Let us consider the following sequence

\[ a_n = \cos \left( \frac{2\pi n}{3} \right) \left( 1 + \frac{1}{n} \right). \]

Then

\[ \lim_{k \to \infty} a_{3k} = \lim_{k \to \infty} \cos \left( \frac{2\pi 3k}{3} \right) \left( 1 + \frac{1}{3k} \right) = \lim_{k \to \infty} \left( 1 + \frac{1}{3k} \right) = 1 \]

and

\[ \lim_{k \to \infty} a_{3k+1} = \lim_{k \to \infty} \cos \left( \frac{2\pi (3k+1)}{3} \right) \left( 1 + \frac{1}{3k} \right) = \lim_{k \to \infty} \cos \left( \frac{2\pi}{3} \right) \left( 1 + \frac{1}{3k} \right) = -\frac{1}{2} \lim_{k \to \infty} \left( 1 + \frac{1}{3k} \right) = -\frac{1}{2} \]

Therefore, the sequence \((a_n)_{n \in \mathbb{N}}\) is a divergent sequence.
Theorem 7 Any sequence has a monotone subsequence
Proof. Let \((a_n)_{n \in \mathbb{N}}\) be a sequence. We will call a peak a term \(a_m\) of the sequence such that \(a_m \geq a_n\) for all \(n \geq m\). For the first case, let us suppose that \((a_n)_{n \in \mathbb{N}}\) has infinitely many peaks. We list those peaks by increasing order of their subscripts

\[ a_{m_1}, a_{m_2}, \ldots, a_{m_k}, \ldots \]

and clearly by construction we have

\[ a_{m_1} \geq a_{m_2} \geq \cdots \geq a_{m_k} \geq \cdots \]

Which is clearly a decreasing subsequence of \((a_n)_{n \in \mathbb{N}}\). For the second case, let us suppose that \((a_n)_{n \in \mathbb{N}}\) has a finite number of peaks (possibly none.) We list them in increasing order of their subscripts as follows

\[ a_{m_1}, a_{m_2}, \ldots, a_{m_k}. \]

Since \(a_{m_k}\) is the last peak, then it is clear \(a_{m_{k+1}}\) is not a peak. Thus, there is \(n_1 > m_k + 1\) such that

\[ a_{m_{k+1}} < a_{n_1}. \]

But \(a_{n_1}\) is not a peak. Thus, there is \(n_2 > n_1\) such that \(a_{n_1} < a_{n_2}\). We may then iterate this process indefinitely to construct an increasing sequence

\[ a_{n_1} < a_{n_2} < a_{n_3} < \cdots < a_{n_k} < \cdots \]

\[ \blacksquare \]
Theorem 8 (The Bolzano-Weierstrass Theorem) Any bounded sequence has a convergent subsequence.

Proof. Let \((a_n)_{n \in \mathbb{N}}\) be a sequence which is also bounded. Now from our previous result, we know that \((a_n)_{n \in \mathbb{N}}\) has a monotone subsequence say \((a_{n_k})_{k \in \mathbb{N}}\). Since \((a_{n_k})_{k \in \mathbb{N}}\) is a bounded sequence (as a subsequence of a bounded sequence) then \((a_{n_k})_{k \in \mathbb{N}}\) must be a convergent subsequence of the parent sequence \((a_n)_{n \in \mathbb{N}}\).
Example 9 Consider the sequence with terms $\cos(n)$. Since this sequence is bounded, then it has a convergent subsequence.