Mathematical Proofs

What is a proof? A proof is a justification of the truth of a statement.

As you write a proof, be sure it is not just a string of symbols. Every step of your proof should be expressed as a complete sentence.

Recall the following:

An "if then" statement is of the form

$$p \Rightarrow q.$$ 

Its negation is \( \sim(p \Rightarrow q) \) and it is equivalent to \( p \land (\sim q) \).

There are many ways to prove an if then statement.

1. Direct proof. Start with \( p \) and get to \( q \).
2. Proof by contradiction. Prove that the negation of \( p \Rightarrow q \) is a contradiction, that is, prove that \( p \land (\sim q) \) is always false.

Here are some examples.
Ex. Using direct proof, prove that if \( n \) is an even integer then \( -5n-3 \) is an odd integer.

Proof: Assume that \( n \) is even. Then there exists \( q \in \mathbb{Z} \) such that

\[
\begin{align*}
n &= 2q, \\
-5n &= -10q, \\
-5n-3 &= -10q-3 \\
&= -10q-4+1 \\
&= 2(-5q-2)+1 \\
&= 2k+1 \quad \text{for } k = -5q-2
\end{align*}
\]

Since \( k = -5q-2 \) is an integer, then \( -5n-3 \) is odd.

Ex. Using proof by contradiction.

Here \( p \) is "\( n \) is an even integer" and \( q \) is "\( -5n-3 \) is an odd integer."

So we want to show that \( p \land (\neg q) \) is always false. In other words, it is a contradiction.

\( n \) is an even integer means there exists \( q \in \mathbb{Z} \) such that \( n = 2q \).

\( \neg q \) means that \( -5n-3 \) is even. In other words, there exists \( k \) such that

\[
-5n-3 = 2k.
\]
Thus,
\[-5n - 3 = -5(2q) - 3 = 2k\]
\[-10q - 3 = 2k\]
\[-10q - 2k = 3\]
\[5q + 2k = -3\frac{1}{2}\]

Not possible since \(5q + 2k \neq 2k\). Thus

the statement \(\sim(p \Rightarrow q)\) is false \(\Rightarrow p \Rightarrow q\) is true.
0.2 A review of Terminology

A right angle. An angle which measures 90°.

Obtuse angle. An angle whose angular measure is greater than 90°.

Acute angle. An angle whose angular measure is less than 90°.

Circle. A circle with center O and radius r is the set of points a distance r away from O.

A chord. A chord of a circle is a line segment with endpoints on the circle.

Diameter. A diameter of a circle is a chord which passes through the center. The diameter may also refer to the length of a diameter.

Radius. A line segment from the center of a circle to the circle. The radius may also refer to the length of a radius.

Arc. A portion of a circle. The angular measure of an arc is the measure of the angle created by the radii which go to the ends of the arc.
Triangle: A set of 3 pts (vertices) together with the 3 lines segment joining them.

Degenerate triangle: A triangle whose 3 vertices are collinear.

Isosceles triangle: A triangle with 2 sides of equal length.

Scalene triangle: A triangle for which no two sides are equal.

Right angle triangle: A triangle with a right angle.

Quadrilateral: A set of 4 points joined by 4 non-intersecting segments.

Square: A quadrilateral whose sides have equal length and whose angles are equal.

Rhombus: A quadrilateral whose sides are equal.

Parallelogram: A quadrilateral whose opposite sides are parallel.

Supplementary angles: 2 angles which together form a straight line.

Adjacent angles: 2 angles which share a common ray.

Complementary angles: 2 angles which together form a right angle.
Vertical angles: 2 opposite angles at the intersection of 2 lines
0.3 Notes on notation

We denote the length of the segment $AB$ with $|AB|$. We denote the segment $AB$ with $\overline{AB}$. We denote the line $AB$ with $\overrightarrow{AB}$. We denote the ray $AB$ with $\overleftarrow{AB}$. We denote the arc $AB$ with $\widehat{AB}$. We denote the triangle $ABC$ with $\triangle ABC$ and its area $|\triangle ABC|$. We denote the measure of the angle $\angle ABC$ with $\angle ABC$. 
Theorem 1.1.1 The Pythagorean theorem.

Suppose that a triangle $\triangle ABC$ has a right angle at $C$, hypotenuse $c$ and sides $a$ and $b$. Then

$$c^2 = a^2 + b^2.$$

Proof

Area of larger square:

$$\left(a + b\right)^2 = a^2 + 2ab + b^2 \quad (1)$$

Area of larger square:

$$4 \left(\frac{bc}{2}\right)^2 + c^2 = b^2 \cdot 2ab + c^2 \quad (2)$$

$(1) = (2) \Rightarrow a^2 + 2ab + b^2 = 2ab + c^2$ 

$\Rightarrow a^2 + b^2 = c^2.$
Theorem 1.1.2 (The converse of the Pythagorean Theorem)

Suppose we are in a geometry where the Pythagorean theorem is valid. Suppose in triangle $\triangle ABC$ we have $a^2 + b^2 = c^2$ then $C$ is a right angle.

Proof

Let $L$ be the line perpendicular to $BC$ passing through $A$. Suppose that $L$ intersects $BC$ at $D$. Let $x = |AD|$, $s = |DC|$. By the Pythagorean theorem, we have that $b^2 = s^2 + x^2$. Using the picture above, notice that by the Pythagorean theorem, we also have that $(a-s)^2 + s^2 = c^2$.

Now, $b^2 = s^2 + x^2 \Rightarrow x^2 = b^2 - s^2$. Thus,

\[
(a-s)^2 + s^2 = (a-s)^2 + (b^2 - s^2) = a^2 - 2as + s^2 + b^2 - s^2 = a^2 - 2as + b^2 = c^2.
\]

By assumption, $a^2 + b^2 = c^2$. Thus $a^2 - 2as + b^2 = a^2 + b^2 = c^2$.

This implies that $2as = 0$. Since $a \neq 0 \Rightarrow s = 0$. As a result, $D = C$ and the triangle is right at $C$. 
Conclusion. Let \( \triangle ABC \) be a triangle. \( c^2 = a^2 + b^2 \) if and only if \( \triangle ABC \) is a right triangle at \( C \).
1.2 The Axioms of Euclidean Geometry

Based on many assumptions made in section 1.1 the results obtained may not be valid.

**Def.** An unquestionable or absolute fact is called an axiom or postulate.

**Euclid's postulates**

1. We can draw a unique line between any 2 pts.
2. Any line segment can be continued indefinitely.
3. A circle of any radius and any center can be drawn.
4. Any 2 right angles are congruent.
5. Given a line \( l \) and a point \( P \) not in \( l \), there exists a unique line \( l_2 \) through \( P \) which does not intersect \( l \).

Notice that these axioms presuppose the following:

* A notion of distance (we are able to measure)
* All pts exist in a 2 dimensional plane.
* A notion of lines, circle, angular measure and congruency.
Def. A distance is a function \( d \) which assigns a positive real number to any pair of pts in the plane.

\[
d : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow [0, \infty)
\]

such that

a) \( d(P, Q) = d(Q, P) \) for all \( P, Q \in \mathbb{R}^2 \)

b) \( d(P, Q) > 0 \)

c) \( d(P, R) \leq d(P, Q) + d(Q, R) \) for all \( P, Q, R \in \mathbb{R}^2 \) (triangle inequality)

We say the distance between \( P \) and \( Q \) is \( d(P, Q) \) and

\[
d(P, Q) = |PQ|.
\]

Def. The circle \( C_p(r) \) centered at \( P \) with radius \( r \) is the set

\[
C_p(r) = \{ Q : |PQ| = r \}.
\]
Ex. Draw a circle of radius 1 centered on (0, 0) on a cartesian plane.

Such circle is called a unit circle.

Def. An isometry of the plane is a map from the plane to itself which preserve distances. That is, if \(f\) is an isometry, then \(f: \mathbb{R}^2 \rightarrow \mathbb{R}^2\) such that

\[ d(P, Q) = d(f(P), f(Q)). \]
Ex. Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) such that
\[
f((x, y)) = (x + 1, y).
\]
Show that \( f \) is an isometry using the Euclidean distance.

Let \( P = (x_1, y_1) \), \( Q = (x_2, y_2) \).

\[
f(P) = (x_1 + 1, y_1), \quad f(Q) = (x_2 + 1, y_2).
\]

\[
d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

\[
d(f(P), f(Q)) = \sqrt{(x_2 + 1 - x_1 - 1)^2 + (y_2 - y_1)^2}
\]

\[
= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.
\]

Thus \( d(f(P), f(Q)) = d(P, Q) \) and \( f \)

is an isometry of the Euclidean plane.
Two sets of points are congruent if there exists an isometry which maps one set to the other.

In particular 2 angles are congruent if there exists an isometry which sends one angle to the other, and we write that

\[ \triangle ABC = \triangle A'B'C' \] if there exists an isometry \( f \) such that

\[ f(A) = A', \]
\[ f(B) = B', \]
\[ f(C) = C'. \]

In fact congruent angles have equal measures.

Now, we will add 3 more axioms to Euclid 5 postulates

6. Given \( P, Q \in \mathbb{R} \) there exist an isometry \( f \) such that \( f(P) = Q \). Such isometry is called a translation.

7. Given a pt \( P \) and \( Q, R \) such that \( |PQ| = |PR| \) there exists an isometry which fixes \( P \) and sends \( Q \) to \( R \). (Reflection) (Rotation)
8. Given any line $l$, there exists an isometry which fixes every point on $l$ but fixes no other points in the plane. (the reflection through $l$)

**Def. Right angle.** Two lines $l_1$, $l_2$ intersect at right angles if any 2 adjacent angles at the pt of intersection are congruent.

![Diagram of right angles and perpendicular lines]

- $d_1 = d_3 = d_2 = d_4$
- $d_4 = d_3$, $d_2 = d_4$, but $d_4 \neq d_2$
- Non perpendicular lines.

**Perpendicular lines**
1.3 SSS, SAS and ASA

Recall that 2 triangles are congruent if their 3 sides are equal (SSS or side-side-side) or if 2 sides are equal and the angle between them are equal (SAS or side-angle-side) or

or if 2 angles and the side between them are equal. (ASA)

**Theorem (SSS)** If the corresponding sides of 2 triangles $\triangle ABC$ and $\triangle A'B'C'$ have equal lengths, then the 2 triangles are congruent.

**Lemma 1.3.2** 2 distinct circles intersect in zero, one, or 2 points. If there is exactly one point of intersection then that pt lies on the line joining the 2 centers.
I = \mathcal{C}_1 \cap \mathcal{C}_2
Suppose that \( C_1 \cap C_2 = \emptyset \) but \( I \neq 0_10_2 \).

Have
1. \( I0_1I = r_1 = \text{radius of } C_1 \)
2. \( I0_2I = r_2 = \text{radius of } C_2 \).

Let \( I' \) be the reflection of \( I \) through line \( 0_10_2 \).
Let \( l \) be the line passing through \( I + 0_10_2 \).

Case 1: Suppose \( l \) intersects inside the segment \( 0_10_2 \) at \( Q \).

By SSS, we have that \( \Delta I0_10_2 \) is congruent to \( \Delta I'0_10_2 \). Thus \( |I0_1| = |I'0_1| = r_1 \) and \( |I0_2| = |I'0_2| = r_2 \).

Thus, \( I' \in C_1 \), \( I' \in C_2 \) \( \Rightarrow I' \in C_1 \cap C_2 \)

As a result, \( C_1 \cap C_2 \) contains the set \( \{I, I'\} \).
we have reached a contradiction. Thus, \( I \notin O_1O_2 \).

**Proposition**: Side-Angle-Side (SAS)

If \( AB = A'B' \), \( BC = B'C' \) and \( \angle B = \angle B' \) then \( \triangle ABC \cong \triangle A'B'C' \). (There is an isometry such that \( f(\triangle ABC) = f(\triangle A'B'C') \).)

**Proof**: (From the book of Euclid)

Since \( \angle B \) is congruent to \( \angle B' \), there exists an isometry \( T \) such that \( T(\angle B) = \angle B' \). Next, if necessary we may apply a reflection to ensure that \( T(\overrightarrow{BA}) < \overrightarrow{B'A'} \) and that \( T(\overrightarrow{BC}) < \overrightarrow{B'C'} \). Next, since \( |BA| = |B'A'| \) and \( |BC| = |B'C'| \) then \( T(A) = A', T(B) = B' \).
Short proofs, we have the followings

Prop: (ASA).

If $LA = LA'$ and $LB = LB'$ and $|AB| = |A'B'|$ then there is an isometry such that $\mathcal{f}(\Delta ABC) = \Delta A'B'C'$.

Prop (SSS)

If $AB = A'B'$, $AC = A'C'$ and $BC = B'C'$ then there is an isometry $\mathcal{f}$ such that $\mathcal{f}(\Delta ABC) = \Delta A'B'C'$ and we write that $\Delta ABC \equiv A'B'C'$.

Def. An isometry is a direct isometry or proper isometry if the image of every clockwise oriented triangle is oriented clockwise.

Ex. A reflection is not a direct isometry.

Let $\mathcal{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $\mathcal{f}(x,y) = (x, -y)$.

Consider the following triangle

![Diagram of triangle with vertices at (0,0), (1,0), (0,1), and (1,-1).]
A triangle $\Delta (0,0) (1,1) (1,0)$ is clockwise oriented, but $\Delta f(0,0) f(1,1) f(1,0)$ is counterclockwise oriented.
Theorem 1.4.1. Let $P$ be a pt not on $l$, and let $Q$ lie on $l$ so that $PQ \perp l$. Let $l_2$ be the line through $P$ which is $\parallel$ to $l$. Then $l_2 \perp PQ$.

Lemma 1.4.2. Let $l$ be a line and $P$ a pt not on $l$, then there exists $Q \in l$ so that $PQ \perp l$.

Theorem 1.4.3. Suppose that $l \perp l_1$, $l \perp l_2$. Then $l_1 \parallel l_2$. 

\[ l \]
\[ \overline{PQ} \]
\[ l_2 \]
\[ l_1 \]
Cor 1.4.4 Suppose a line \( l \) intersects 2 other lines \( l_1, l_2 \) so that the opposite angles are equal. Then \( l_1 \parallel l_2 \).

Cor 1.4.5 (Euclid's Axiom 5) Suppose a line \( l \) meets 2 other lines \( l_1, l_2 \) so that the sum of the angles on one side is less than 180°. Then \( l_1 \) and \( l_2 \) meet a 1st on that side.
Theorem 1.4.6  The 3 angles in a triangle sum up to 180°.

\[ \alpha + \beta + \gamma = 180°. \]

**Definition (Exterior angle)**

The exterior angle at A in \( \triangle ABC \) is one of the angles adjacent to \( \angle BAC \) at the intersection of the lines \( AB \) and \( AC \).

\[ \alpha \] is the exterior angle at A.
Corollary The exterior angle at A is the sum of the other interior angles.

Proof

We want to show that \( \gamma = \gamma + \beta \).

\( \eta + \alpha = 180 \) and \( \alpha + \beta + \gamma = 180 \).

Thus \( \eta + \alpha = \alpha + \beta + \gamma \Rightarrow \eta = \beta + \gamma \)

Theorem (Pons Asinorum) the base angles of an isosceles triangle are equal.
16 The Star Trek Lemma

Let ABC be 3 pts on a circle centered at O.
We call $\angle BAC$ an inscribed angle.

The angular measure of the arc BC is the measure of the angle $\angle BOC$.
We say $\angle BAC$ subtends $\widehat{BC}$. 
The measure of an inscribed angle is half of the angular measure of the arc it subtends.

\[ \alpha = 2 \beta \]
Theorem 1.6.1 (Star Trek lemma)
the measure of an inscribed angle is half of the angular measure of the arc it subtends

Proof

Let us suppose that \( \angle BAC \) is acute.
We have that \( |OB| = |OA| = |OC| \) = radius.
Consider, the line \( \overrightarrow{OA} \), and let \( D \in \overrightarrow{OA} \cap \overrightarrow{CE} \).

Notice that \( \angle BAO = \angle LOBA \) since the triangle \( OBA \) is isosceles.

Also,
\[ \gamma + \delta + \beta = 180 \Rightarrow \gamma + 2\delta = 180. \]
\[ \gamma' + \delta = 180 \Rightarrow \gamma' = 180 - \delta = \gamma. \]
\[ \gamma' + \alpha + \beta' = 180 \Rightarrow \gamma' + 2\alpha = 180. \]
We have that
\[ \alpha + \alpha' + \beta + \beta' = 180. \quad (1) \]
\[ \alpha + \beta + \gamma = 180 \quad (2) \]
\[ \alpha' + \beta' + \gamma' = 180 \quad (3) \]

Thus \[ \alpha + \beta + \gamma = \alpha' + \beta' + \gamma' \quad (\ast) \]

Since (1) implies that \[ \alpha' + \beta' = 180 - \alpha - \beta. \]

(\ast) becomes
\[ \alpha + \beta + \gamma = \gamma' + 180 - \alpha - \beta. \]
\[ \gamma' + 180 - \gamma = \alpha + \beta + \alpha + \beta \]
\[ \gamma' + 180 - \gamma = 2\alpha + 2\beta. \]
\[ \gamma' + \gamma = 2\alpha + 2\beta \]
\[ \alpha + \gamma = 2\alpha + 2\beta \]
\[ \gamma = \alpha + \beta. \]

Thus, \[ \gamma = \alpha + \beta \Rightarrow \angle BOD = \angle BAD \]

Using a similar proof (homework) we can show that \[ \angle DOC = \angle DAB. \]

\[ \angle BOC = \angle DOC + \angle BOD \]
\[ = \angle DAB + \angle DAB \]
\[ = \angle (\angle DAB + \angle DAB) \]
\[ = \angle DAB. \]
Ex. Suppose that $\angle ABC$ is a right angle inscribed in a circle. Prove that $AC$ is a diameter.

\[
\theta = \frac{1}{2} \angle AOC
\]
\[
= \frac{1}{2} (180^\circ)
\]
\[
= 90^\circ
\]
If $|AB| = |AC| = |BC|$ what is the angle at $D$?

$\theta = 60$ since

$\triangle ABC$ is equilateral, and

$\angle ABC = \angle ADC$
Ex 1.33

Suppose that AT is the line segment that is tangent to a circle. Prove that LATB is half the measure of the arc TB

\[
(90 - \alpha) + 90 + \beta = 180
\]

\[
\alpha - \beta = 0
\]

\[
\alpha = \beta
\]

Thus \( \alpha = \frac{1}{2} \) measure of arc TB
Bx 1.34 Suppose that 2 lines intersect at P inside a circle and meet the circle at A and A', and at B at B'. Let \( \alpha, \beta \) be the measures of the arcs \( A'B' \) and \( AB \) respectively.

Show that \( \angle APB = \frac{\alpha + \beta}{2} \)

Let \( \alpha = \text{measure of arc } A'B' \)
\( \beta = \text{II of arc } AB \)

Now, we have

\[
360 = 2\eta + 2 \left( 180 - \frac{1}{2} (\alpha + \beta) \right)
\]

\[
\eta = \frac{\alpha + \beta}{2}
\]
1.7 Similar Triangles

**Theorem 1.7.1** Let $B'$ and $C'$ be on the respective sides $AB$ and $AC$ of $\triangle ABC$. Then $B'C' \parallel BC$ if
\[
\frac{|AB'|}{|AB|} = \frac{|AC'|}{|AC|}
\]

We will not prove this theorem.

**Theorem** Let $\triangle ABC$ be a triangle. Let $B' \in AB$, and let the line through $B'$ and $\parallel BC$ intersect $AC$ at $C'$. Then
\[
\frac{|AB'|}{|AB|} = \frac{|AC'|}{AC}
\]
Theorem 1.7.3 Let $B'$, $C'$ be pts on sides $AB$ and $AC$ of \( \triangle ABC \). Suppose
\[
\frac{|AB'|}{|AB|} = \frac{|AC'|}{|AC|}
\]
Then \( B'C' \parallel BC \).

Proof Let \( l \) be the line through $B'$ and $\parallel BC$.

\[
\text{such} \ l \cap AC = \{O''\}.
\]
According to the previous theorem, we have
\[
\frac{|AB'|}{|AB|} = \frac{|AC''|}{|AC|} = \frac{|AC'|}{|AC|}
\]
Thus, \( |AC''| = |AC'| \). Since \( \overline{C''} \subset \overline{AC} \). Then \( c'' = c' \).

This is a segment.
Def: We say that two triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar if their angles are congruent.

We write $\triangle ABC \sim \triangle A'B'C'$.

Cor: If $\triangle ABC \sim \triangle A'B'C'$ then
\[
\frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|} = \frac{|B'C'|}{|BC|}.
\]

Proof: Homework.

Hint: Find an isometry $f$ such that
\[f(A) = A, \quad f(B') \in AB, \quad f(C') \in AC.\]

Ex: Suppose that $\triangle ABC$ is similar to $\triangle A'B'C'$. Show that
\[
|\triangle A'B'C'| = \left(\frac{|A'B'|}{|AB|}\right)^2 |\triangle ABC|.
\]
\[ |\Delta ABC| = \frac{h}{a} |BC| \]

\[ |\Delta A'B'C'| = \frac{h'}{a'} |B'C'| \]

Since \( \frac{|B'C'|}{|BC|} = \frac{|A'B'|}{|AB|} \),

\[ |B'C'| = \frac{|BC| \cdot |A'B'|}{|AB|} \]

Thus,

\[ |\Delta A'B'C'| = \frac{h'}{a'} \frac{|BC| \cdot |A'B'|}{|AB|}, \quad |BC| = 2 \cdot \frac{|\Delta ABC|}{h} \]

\[ = \frac{h'}{2a} \cdot \frac{2|\Delta ABC|}{h} \cdot \frac{|A'B'|}{|AB|} \]
\[ |\Delta A'B'C'| = \frac{h'}{2} \left( \frac{|\Delta ABC|}{|AB|} \right) \left( \frac{|A'B'|}{|AB|} \right) \]

\[ = \frac{h'}{h} \frac{|A'B'|}{|AB|} |\Delta ABC| \]

\[ = \left( \frac{|A'B'|}{|AB|} \right)^2 |\Delta ABC| \]
Application

\[ |\Delta ABC| = \frac{1}{2} \]
\[ |\Delta A'B'C'| = \frac{1}{8} \]

Notice that

\[ |\Delta A'B'C'| = \frac{1}{8} = \frac{1}{2} \cdot \frac{1}{4} = |\Delta ABC| \cdot \frac{1}{4} \]

\[ \left( \frac{|A'B'|}{|AB|} \right)^2 = \left( \frac{1}{2} \right)^2 = \frac{1}{4} \]

Thus

\[ |\Delta A'B'C'| = \left( \frac{|A'B'|}{|AB|} \right)^2 |\Delta ABC| \]
Review on Vectors

Consider $A = (a_1, a_2), B = (b_1, b_2)$, two points in the plane.

The vector $\vec{AB}$ has for component

$$\vec{AB} = \langle b_1 - a_1, b_2 - a_2 \rangle$$

Ex Given $A = (1, 2), B = (2, 3)$,

Find the components of $\vec{AB}$ and $\vec{BA}$.

$$\vec{AB} = \langle 1, 1 \rangle, \quad \vec{BA} = \langle -1, -1 \rangle$$

Notice that $\vec{AB} = -\vec{BA}$.

Given 2 vectors $\vec{u}, \vec{v}$ with

$$\vec{u} = \langle u_1, u_2 \rangle, \quad \vec{v} = \langle v_1, v_2 \rangle$$
The inner product of $\vec{u},\vec{v}$ is

\[
\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2
\]

Ex: Let $\vec{u} = \langle 1, 2 \rangle$, $\vec{v} = \langle 2, -1 \rangle$

\[
\vec{u} \cdot \vec{v} = \langle 1, 2 \rangle \cdot \langle 2, -1 \rangle = 2 - 2 = 0.
\]

- The magnitude or norm of $\vec{u} = \langle u_1, u_2 \rangle$ is given by

\[
||\vec{u}|| = \sqrt{u_1^2 + u_2^2}
\]

- Also, if $\theta$ is the angle between $\vec{u}, \vec{v}$

\[
\vec{u} \cdot \vec{v} = ||\vec{u}|| ||\vec{v}|| \cos \theta
\]

\[
\theta = \arccos \left( \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| ||\vec{v}||} \right)
\]
Find the angle between the vectors $\vec{u} = \langle 1, 0 \rangle$ and $\vec{v} = \langle 1, 1 \rangle$

$$\alpha = \arccos \left( \frac{1 + 0}{\sqrt{2}} \right)$$

$$= \arccos \left( \frac{\sqrt{2}}{2} \right)$$

$$= \frac{\pi}{4}$$
Some facts
Consider the figure below

The slope of the line passing through \( AB \) is \( \tan \alpha \). Since \( \alpha = \arccos \left( \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}| |\vec{AC}|} \right) \)

thus \( \tan \alpha = \tan \left( \arccos \left( \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}| |\vec{AC}|} \right) \right) \)
Ex Find the equation of the line bisecting \( \alpha \)

\[
\overrightarrow{OA} = \langle 1, 0 \rangle, \quad \overrightarrow{OB} = \langle \frac{1}{2}, 1 \rangle
\]

\[
\overrightarrow{OB} \cdot \overrightarrow{OA} = \frac{1}{2} \quad \Rightarrow \quad \cos \alpha = \frac{\frac{1}{2}}{1 \cdot \sqrt{\frac{1}{4} + 1}}
\]

\[
= \cos \alpha = \frac{\frac{1}{2}}{1 \cdot \sqrt{\frac{5}{4}}}
\]

\[
= \frac{\frac{1}{2}}{\sqrt{\frac{5}{4}}} = \frac{1}{2} \cdot \frac{2}{\sqrt{5}} = \frac{1}{\sqrt{5}}
\]
\[ \cos \theta = \frac{1}{\sqrt{5}} \implies \alpha = \arccos \left( \frac{1}{\sqrt{5}} \right) \]

The slope of the line bisecting \( \alpha \) is

\[ \tan \left( \frac{\alpha}{2} \right) = \tan \left( \frac{1}{2} \arccos \left( \frac{1}{\sqrt{5}} \right) \right) \]

Thus the equation of the line bisecting \( \alpha \) is

\[ y = \tan \left( \frac{1}{2} \arccos \left( \frac{1}{\sqrt{5}} \right) \right) x \]
The power of a point is a real number that reflects the relative distance of a given point from a given circle.
1.8 Power of the Point

Theorem 1.8.1 (Power of the Point)

\[ |PQ||PQ'| = |PR||PR'| \]

Proof (Homework)
Proof

By star Trek Lemma \( \alpha = \beta \) and \( \alpha' = \beta' \).

Thus the triangles \( \triangle RKQ'P \) and \( \triangle QR'P \) are similar triangles. Thus

\[
\frac{|PQ'|}{|PR|} = \frac{|PR|}{|PQ|}.
\]

Thus

\[
|PQ'| |PQ| = |PR| |PR'|
\]
We call the product

$$
\Pi(P) = \pm |PQ||PQ'|$$

the power of the point with respect to the circle \( \Gamma \).

\[
\Pi(P) = |PQ||PQ'| \text{ if } P \text{ is outside } \Gamma
\]

\[
\Pi(P) = -|PQ||PQ'| \text{ if } P \text{ is inside } \Gamma.
\]

**Remark**  If \( \Gamma \) has for center \( O \) and radius \( r \), then

\[
\Pi(P) = |OP|^2 - r^2.
\]

**Proof**

\[
\Pi(P) = -|PQ||PQ'| = -(r-|OP|)(r+|OP|) = |OP|^2 - r^2.
\]
Ex: Let $C$ be a unit circle

$$x^2 + y^2 = 1.$$ 

Let $P = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$\Pi(P) = |OP|^2 - r^2$$

$$= \left(\frac{1}{4} + \frac{1}{4}\right) - 1$$

$$= \frac{1}{2} - 1 = \boxed{-\frac{1}{2}}$$
Let \( l \) be the line passing through \( P \) and \( \perp \) to the \( x \)-axis.

Let \( Q, Q' \) be the pts of intersection of \( l \) and \( \mathbb{C} \).

\[
\begin{cases}
  x = \frac{1}{a} \\
  x^2 + y^2 = 1
\end{cases}
\Rightarrow
y^2 = 1 - \frac{1}{y} = \frac{3}{4}
\Rightarrow
y = \pm \frac{\sqrt{3}}{2}
\]

\[Q = \left( \frac{1}{a}, \frac{\sqrt{3}}{a} \right), \quad Q' = \left( \frac{1}{a}, -\frac{\sqrt{3}}{2} \right)\]

\[|PQ| = \sqrt{0 + \left( \frac{\sqrt{3}}{a} - \frac{1}{2} \right)^2} = \frac{\sqrt{3}}{a} - \frac{1}{2}\]

\[|PQ'| = \frac{\sqrt{3}}{a} + \frac{1}{2}\]

\[\Pi(P) = -|PQ||PQ'| = -\frac{(\sqrt{3} - 1)(\sqrt{3} + 1)}{4} = -\frac{3 - 1}{4} = \frac{1}{2} = \sqrt{\frac{1}{4}}\]
Ex 1.50

$|AP| = 2, \ |AB| = 6, \ |PC| = 3$

What is $|PD|$?

$3x = 8 \implies x = \frac{8}{3}$
**Def:** The radical axis of 2 circles \( P \) and \( P' \) is the set of points \( P \) with the property that the powers of the point \( P \) with respect to both \( P, P' \) are equal.

**Theorem:** The radical axis is a line. (The proof is an exercise)

**Ex:** Suppose that 2 circles intersect at \( A \) and \( B \).

Prove that the radical axis of the 2 circles is \( AB \)

Since the radical axis \( l \) is a line, it suffices to find 2 pts on the line.

**Claim:** \( A, B \) belong to the radical axis

\[
\Pi_1(A) = 10_1 A - \eta_1^2 = 0
\]
\[
\Pi_2(A) = 10_2 A - \eta_2^2 = 0
\]
Thus \( \Pi_1(A) = \Pi_2(A) = 0 \)

Similarly \( \Pi_1(B) = \Pi_2(B) = 0 \).
There is a unique line passing through the pts A, B, thus the line AB is the radical axis. This completes the proof.
Theorem 1.10.1

The angle bisectors of a triangle intersect at a common point I called the incenter, which is the center of the unique circle inscribed in the triangle called the incircle.
Construction of Incircle

I = Incenter
Example: Find the coordinates of the Incenter and Incircle of the triangle passing through \( A, B, C \).
\[ A = (0,0) \quad B = (2,0) \quad C = (3,4) \]

We need to find angles \( \alpha, \beta, \gamma \).
Consider the vectors \( \overrightarrow{AC} \) and \( \overrightarrow{AB} \)

\[ \overrightarrow{AC} = \langle 3, 4 \rangle, \quad \overrightarrow{AB} = \langle 2, 0 \rangle \]

\[ \overrightarrow{AC} \cdot \overrightarrow{AB} = |AC| \cdot |AB| \cdot \cos \alpha \]

\[ b = \sqrt{9 + 16} \cdot \sqrt{4} \cdot \cos \alpha \]

\[ = 5 \cdot 2 \cdot \cos \alpha \]

\[ \cos \alpha = \frac{b}{50} = \frac{3}{5} \]

Now consider the vectors \( \overrightarrow{BA}, \overrightarrow{BC} \)

\[ \overrightarrow{BA} = \langle -2, 0 \rangle, \quad \overrightarrow{BC} = \langle 1, 4 \rangle \]

\[ \overrightarrow{BA} \cdot \overrightarrow{BC} = |BA| \cdot |BC| \cdot \cos \beta \]

\[ -2 + 0 = 2 \cdot \sqrt{1 + 16} \cdot \cos \beta \]
\[ \cos \theta = \frac{1}{11} \]

\[ \alpha/2 = 0.46 \text{ rad} \]

\[ \beta/2 = 0.66 \text{ rad} \]

Equation of bisector of \( \alpha \) is

\[ y = \tan \left( \frac{\alpha}{2} \right) x = \frac{1}{2} x \]

Equation of line bisector of \( \beta \) is

\[ y = -\tan \left( \frac{\beta}{2} \right) (x-2) \]

\[ = -0.78 (x-2) \]
To find the coordinates of the incircle we solve the system

\[
\begin{align*}
    y &= \frac{1}{2} x \\
    y &= -0.78(x - 2)
\end{align*}
\]

\[I = (1.21, 0.61)\]
In $\Delta ABC$, the excenter $I_a$ is the point of intersection of the interior angle bisector at $A$ and the exterior angle bisector at $B$ and $C$. The excenters $I_b, I_c$ are defined similarly.

In fact $I_a$ is the center of the circle tangent to the side $BC$, tangent to the extended sides $AB, AB$ and lies outside $\Delta ABC$. This circle is called an excircle.
How to plot $I_a$ and its corresponding excircle
let $r$ be the inradius of the incircle.
Let $r_a, r_b, r_c$ be the exradii.

Define $s = \frac{1}{2}(a+b+c)$
the semiperimeter of $\triangle ABC$

\textbf{Theorem 1.10.2}

\[ |\triangle ABC| = rs \]

\textbf{Proof}
Consider $\Delta BCI$

\[ |\Delta BCI| = \frac{1}{2} |\Delta ABC| \cdot |BD| = \frac{1}{2} ar. \]

Similarly,

\[ |\Delta CIA| = \frac{1}{2} |\Delta ACD| r = \frac{1}{2} br \]
\[ |\Delta AIB| = \frac{1}{2} |\Delta ABD| r = \frac{1}{2} cr \]

\[ |\Delta ABC| = |\Delta BCI| + |\Delta CIA| + |\Delta AIB| = \frac{1}{a} (ar + br + cr) \]
\[ = \frac{1}{a} (a + b + c) r = Sr. \]
Theorem 10.3 (The Law of Cosines)

For any triangle \( \triangle ABC \), we have

\[ c^2 = a^2 + b^2 - 2ab \cos C \]

Here is a proof.
\[ c^2 = |AD|^2 + |DB|^2 \\
= (b \sin C)^2 + |DB|^2 \\
= (b \sin C)^2 + (a - b \cos C)^2 \\
= b^2 \sin^2 C + a^2 - 2ab \cos C + b^2 \cos^2 C \\
= b^2 + a^2 - 2ab \cos C. \]
Theorem 1.10.4 (Heron's Formula)

For any triangle \( \triangle ABC \),

\[
|\triangle ABC| = \sqrt{s(s-a)(s-b)(s-c)}
\]

Proof (sketch)

\[
|\triangle ABC| = \frac{1}{2} ab \sin C
\]

\[
\cos C = \frac{a^2 + b^2 - c^2}{2ab}
\]

\[
|\triangle ABC| = \frac{1}{2} ab \sqrt{1 - \cos^2 C} = \frac{1}{2} ab \sqrt{1 - \left(\frac{a^2 + b^2 - c^2}{2ab}\right)^2}
\]

(complete for homework):

\[
|\triangle ABC| = \sqrt{s(s-c)(s-b)(s-a)}
\]

where \( s = \frac{1}{2} (a+b+c) \)
Ex: What is the area of \( \Delta ABC \) if \( a = 3 \), \( b = 5 \), \( c = 6 \)?

\[
\begin{align*}
S &= \frac{1}{2} (3+5+6) \\
&= \frac{1}{2} (14) = 7 \\
A &= \sqrt{7(7-3)(7-5)(7-6)} \\
&= \sqrt{7 \cdot 4 \cdot 2} \\
&= 2 \sqrt{14}
\end{align*}
\]
Chapter 3  Constructions Using a Compass and Straightedge

In this chapter, we will investigate constructions using only a straightedge and compass.

3.1 The Rules

0. We start with 2 distinct pts in the plane
1. We can draw a line through any two already constructed points
2. We can draw a circle with center on already constructed pt and through another already constructed pt
3. We can construct pts at the intersection of two distinct constructed lines, two distinct constructed circles or a constructed line and a constructed circle.
3.2 Some Examples
Def: A figure is **constructible** if we can construct it by applying step 0 and a finite number of steps 1-3 as outlined above. The sequence of steps is called a construction.

### 3.2 Some Examples

**Ex**: We can construct an equilateral triangle.
Theorem: We can construct an equilateral triangle.

\[ \triangle OPQ \] is an equilateral triangle. Finish the proof for homework.
Theorem: We can construct a square that is a $\sqrt{2} \times \sqrt{2}$ if we start with base pts 0, P such that |OP| = 1.

1. Draw $C_P(10P1)$
2. Draw $C_Q(10R1)$
3. Draw $C_R(10R1)$
4. $O S R S'$ is a square.
   In fact, it is a $\sqrt{2} \times \sqrt{2}$ by the Pythagorean theorem.
Theorem 3.2.2  We can construct a regular hexagon.

1. We start with the base pts O, P
2. Construct C_p (10P1)

3. Construct C_{0|OP1}
4. Construct C_{Q|OP1}
3.3 Basic Results

Lemma

\(\text{If } \angle CAB \text{ is constructible, we can bisect it.}\)

Step 1: Construct \(C_A(IAB_1)\), let \(c' = C_A(IAB_1) \cap AC\)

Step 2: \(C_B(IBC_1)\) and \(C_c'(IBC_1)\).
Lemma 8.8.2. We can construct the perpendicular bisector of any arbitrary line segment.

Step 1. Construct \( C_A(AB) \)

Step 2. Construct \( C_B(AB) \)

Step 3. Line through the intersection of \( C_A(AB) \) and \( C_B(AB) \).
Lemma 3.33  Suppose \( \overline{AB}, \overline{BC} \) are constructed. Then we can construct \( \overline{CA} \) (i.e. \( \overline{BC} \)). (We have 2 cases to consider for the proof.)

Proof: If \( \overline{AB} \neq \overline{BC} \), construct the midpt. of \( \overline{AB} \) and construct \( \overline{CB} \).
Assume that $|AB| \leq 2|BC|$

We construct the midpoint of $AB$, $M$.

Let $D \in C_B |BC| \cap AB$

$D' \in C_M(|MD|) \cap AB$

Notice that $|AD'| = a$ since

$|MS'| = a + \frac{c}{2} = |MA| + |AD'| = \frac{c}{2} + a \Rightarrow |AD'| = a$

(See below)
Lemma 3.3.4 Given a line $AB$ and a point $C$, we can construct the line through $C$ which is parallel to $AB$.

Step 1: We construct $C_B(1AC) = \Gamma_1$
Step 2: We construct $C_C(1AB) = \Gamma_2$
Step 3: Let $A' \in \Gamma_1 \cap \Gamma_2$
Step 4: $A'C \parallel AB$. 

$l \parallel AB$
Homework 3.1 Drawing a square with sides of length one.

\[ 2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \frac{\sqrt{2}}{2} \]

Start with base pts 0, P such that \( |OP| = \frac{\sqrt{2}}{2} \)
Homework: Ex 3.2
Construct \( C_A (|AB|) \), \( C_B (|AB|) \) the line through the intersections of the circles is the radical axis and is perpendicular to \( AB \).
Homework Ex 3.6
3.4 The Algebra of Constructible Lengths

From now on, we may assume that the distance between base pts is equal to 1. We say that \( a \) is constructible \( \iff \) there exist two constructible pts \( P, Q \) such that \( |PQ| = a \).

**Lemma** Suppose that \( a, b \) are constructible, then there exist constructible points \( A, C, C' \) so that \( |AC| = a + b \), \( |AC'| = a - b \).

**Proof** Let \( A, A', B, B' \) be constructed pts such that \( |AA'| = a \), \( |BB'| = b \).

To see that

1. Draw line \( AA' \)
2. Draw circle \( C_{A'}(b) \)

Let \( C' \) be on the segment \( AA' \) and \( C \) on the other side (see fig)
Clearly $|AC'| = a - b$ and $|AC| = a + b$
Lemma: If $A, A'$ are constructed points, we can construct the perpendicular line through either $A$ or $A'$.

Step 1: Start with base pts $A, A'$.

Step 2: Construct $C_A(IAA'/I)$, construct $B \in AA' \cap C_A(IAA')$.

Step 3: Construct $C_A(1ABI), C_B(1ABI)$.

Let $D \in C_A(1ABI) \cap C_B(1ABI)$. 
Lemma 3.4.2 Suppose $a$ and $b$ are constructible then $ab$ is a constructible length.

Proof On a constructible line with constructible point $A$.

1. Find $B$ such that $|BA| = 1$ on the constructible line
2. Find $A'$ such that $|AA'| = a$ on the constructible line
3. Find $B'$ on the perpendicular to $AA'$ at $B$ such that $|BB'| = b$.
4. Let $C$ be at the intersection of $AB'$ and the line perpendicular to $AA'$ at $A'$.

Notice that the triangle $ABB'$ and $\triangle AA'C$ are similar. Thus

$$\frac{b}{1} = \frac{|A'C|}{a} \implies |A'C| = ab$$
Lemma 3.4.3  Suppose that \( q \) is a constructible length. Then \( \frac{1}{q} \) is constructible.

Proof  (See fig on the next page)

On a constructible line with constructible pt \( A \)

Step 1  Find \( B \) on the constructed line such that \( |BA| = 1 \)

Step 2  Find \( A' \) on the constructed line such that \( |AA'| = q \)

Step 3  Find \( C \) on the perp to \( AA' \) at \( A' \) such that \( |A'C| = 1 \)

Step 4  Find \( B' \) be at the intersection of \( AC \) and the perp to \( AA' \) at \( B \).

Notice that \( \triangle ABB' \) and \( \triangle AA'C \) are similar and

\[
\frac{1}{q} = \frac{|BB'|}{1} \quad \Rightarrow \quad |BB'| = \frac{1}{q}
\]
Ex. Construct \( \frac{2}{3} \)

\[
\frac{2}{3} = \frac{n}{1} \implies n = \frac{2}{3}
\]
Cor 3.4.4 If \(a\) is a positive rational number then \(a\) is constructible length.

Proof If \(a\) is a rational number, then there exist integers \(p, q\) such that \(\gcd(p, q) = 1\) and
\[
a = \frac{p}{q} = p \cdot \frac{1}{q}
\]

We can construct \(\frac{1}{q}\) by assumption, and
\[
q = (1 + 1 + \cdots + 1) \quad \text{q times}
\]

So we can construct \(2, 3, 4, \ldots, q\).

We can construct \(p\) as well.

We can construct \(\frac{1}{q}\).

By Lemma 3.4.2 we can construct \(p \cdot \frac{1}{q} = \frac{p}{q}\).

Thus the length \(\frac{p}{q}\) is constructible.
Lemma 3.45  Suppose that $a$ is a constructible length. Then $\sqrt{a}$ is a constructible length.

Proof
1. Assume $A, A'$ are constructed such that $|AA'| = a$.
2. Construct $B$, a distance $a$ away from $A$ on $AA'$ on the opposite side of $A$ from $A'$.
3. Let $O$ be the midpoint of $A'B$.
4. Construct $C_0(10A'1)$, and the perpendicular at $A'B$ through $A$ is denoted $\ell$.
5. Let $L \equiv LC_0 \cap C_1C' \cap C_0(10A'1) \cap \ell$.

Notice that $|AC| = |AC'|$ (explain why).

By the power of the point, we obtain

$|AC| \cdot |AC'| = 10 \Rightarrow |AC|^2 = a$

$\Rightarrow |AC| = \sqrt{a}$. 
Ex. Construct $\sqrt{5}$

See below $x^2 = 5 \Rightarrow x = \sqrt{5}$
Chapter 6

Hyperbolic geometry

It is possible to construct a model of Euclidean geometry where lines look like circles.

Consider the sphere \( x^2 + y^2 + (z-1)^2 = 1 \) \((S)\)

and the plane \( z = 0 \) \((xy\text{-plane})\) \((P)\)
We define the function
\[ f : P \rightarrow S \] such that
\[ f(x_0, y_0, 0) = SN \text{ line} (PN) \]

Consider the following line on \( P \):
\[ \{ (x, y, z) : y = 4, z = 0 \} = \{ (x, 4, 0) : x \in \mathbb{R} \} \]

Let \( P_0 = (t, 4, 0) \).
\[ \overrightarrow{P_0N} = \langle -t, -4, 2 \rangle \]

Equation of line \( P_0N \) is
\[ \begin{align*}
  x &= -t \cdot s \\
  y &= -s \\
  z &= 2s + 1
\end{align*} \quad s \in \mathbb{R} \]

We now compute the intersection with
\[ (-t \cdot s)^2 + (-s)^2 + (2s + 1)^2 = 4 \] (solve for \( s \))
\[ S = -\frac{4}{5+t^2} \quad \text{for } t \in \mathbb{R}. \]

\[
\left( \frac{u^1}{u^2} \right) \left( \frac{4}{5+u^2} \right) \left( \frac{-8}{5+u^2} + 1 \right)
\]

With this model, lines are circles passing through the North pole.
Now, we will define a new geometry called Hyperbolic geometry following these set of axioms.

1. We can draw a unique line segment in between 2 pts
2. Any line segment can be continued indefinitely.
3. A circle of any radius and center can be drawn.
4. Any 2 right angles are congruent.
5. Given any line $l$ and $P \neq l$, there exist 2 distinct lines $l_1, l_2$ through $P$ which do not intersect $l$.
6. Given 2 pts $P, Q$, there exists an isometry $f$ such that $f(P) = Q$.
7. Given $P, Q, R$, $|PQ| = |PR|$ there exists an isometry which fixes $P$ and sends $Q$ to $R$.
8. Given any $l$, there exists a map which fixes every point in $l$ and fixes no other points.
6.2 Results from Neumal Geometry

What are some results we can prove without using either version of the 5th axiom?

Lemma. In any triangle ΔABC we have

\[ A + B < 180^\circ \]

Proof. Suppose \( A + B > 180^\circ \).

Construct ΔABC' so ΔABC = ΔBAC

Case 1. If \( A + B = 180^\circ \)

Then \( \angle CAC' = 180^\circ \) and \( \angle CBC' = 180^\circ \).

Thus lines CA, CB intersect at C' that is a contradiction of the first axiom.
Case 2  \( A + B > 180^\circ \)

The ray CA enters the triangle \( \triangle ABC' \) at A and must intersect the segment BC'.

Also the ray CB must intersect AC'.

Clearly they must meet. Thus the line intersects twice. That would be a contradiction.

Conclusion  \( A + B < 180^\circ \)

Lemma 6.2.4  Given \( \triangle ABC \), there exists a triangle \( \triangle A'B'C' \) such that \( c' = \frac{1}{2} c \).

and \( A' + B' + c' = A + B + C \)

Proof  Let D be the midpoint

![Diagram of triangle ABC with rays and points labeled E, D, and H]
7. The Point-care Models of Hyperbolic geometry

7.1 The Point-care Upper Half plane Models

Planar representation of the complex numbers

Any complex number \( z \) is of the form \( z = x + iy \), where \( i \) is the imaginary unit.

We identify \( z = x + iy \) with the point of coordinates \((x, y)\)

and the Point-care upper half plane model of hyperbolic geometry is the set of points \( \mathcal{H} = \{ x + iy : y > 0 \} \)


together with the arc length element
\[ ds = \frac{\sqrt{dx^2 + dy^2}}{y} \quad \text{(More later)} \]

Ex. Express the equations of the Euclidean line \( ax + by + c = 0 \) and the Euclidean circle \( (x-h)^2 + (y-k)^2 = r^2 \) in terms of complex coordinates \( z = x + iy \).

Put \( \overline{z} = x - iy \).

\[ x = \frac{z + \overline{z}}{2}, \quad y = \frac{z - \overline{z}}{2i}, \quad x^2 + y^2 = |z|^2. \]

\[ ax + by + c = 0 \implies a \left( \frac{z + \overline{z}}{2} \right) + b \left( \frac{z - \overline{z}}{2i} \right) + c = 0 \]
\[ \implies a \left( \frac{z + \overline{z}}{2} \right) - b \left( \frac{z - \overline{z}}{2} \right)i + c = 0 \]
\[ \implies a(z + \overline{z}) - bi(z - \overline{z})i + 2c = 0. \]
\[(x-h)^2 + (y-k)^2 = r^2\]

Put \( z = x + iy \)

\[ u = h + ik \]

\[ z - u = (x-h) + i(y-k) \]

\[ |z-u|^2 = (x-h)^2 + (y-k)^2 \]

\[ |z-(h+ik)|^p = r^2 \]

\[ |z-(h+ik)| = r \]

**Exercise**

Let

\[ S = \{ z \in \mathbb{C} \mid |z| = 1 \} \]

be the unit circle in \( \mathbb{C} \).

Let \( A \) be a Euclidean circle in \( \mathbb{C} \) with center \( re^{i\theta} \), \( r > 1 \) and radius \( s \):

Show that \( A \) is perpendicular to \( S \) if and only if

\[ s = \sqrt{r^2 - 1} \]
7.2 Vertical (Euclidean Lines)

Let \( \vec{r}(t) = (x(t), y(t)) \) be a piecewise smooth curve between \( \vec{r}(t_0) \) and \( \vec{r}(t_1) \).

Recall that the arclength of this curve in Euclidean space is

\[
s = \int_{t_0}^{t_1} \sqrt{x'(t)^2 + y'(t)^2} \, dt
\]

or

\[
s = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]
Example 1: Euclidean length

Let \( \mathbf{r}(t) = (1, t) \) \( t \in [1, 2] \)

\( \mathbf{r}(1) = (1, 1), \mathbf{r}(2) = (1, 2) \)

The Euclidean length of this curve is

\[
\int_1^2 \sqrt{1 + 1} \, dt = \left. t \right|_1^2 = 1.
\]
Now, suppose that \( \vec{r}(t) = (x(t), y(t)) \) represents a curve in the Pointcare' Upper half plane.

\[
S = \int_{t_0}^{t_1} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \, dt
\]

**Ex. 2** The Pointcare' arclength

Let \( \vec{r}(t) = (t, t) \), \( t \in [1, 2] \)

\[
S = \int_{1}^{2} \frac{1}{t} \, dt
\]

\[
= \ln t \bigg|_{1}^{2}
\]

\[
= \ln(2) - \ln(1)
\]

\[
= \ln(2).
\]

Notice that this curve is longer than the example given in ex. 1.
Ex: What is the distance between $1+i$, $2+3i$ in the upper half plane

$A = (1,1), \ B = (2,3)$

$\overrightarrow{AB} = \langle 1,2 \rangle$

Let $\overrightarrow{r}(t)$ be the line through $A, B$

$\overrightarrow{r}(t) = (1+t, 2t+b)$

We want $\overrightarrow{r}(1) = (1,1) \Rightarrow (1+a, 2t+b) = (1,1)$

$\Rightarrow a = 0, \ b = -1$

$\overrightarrow{r}(t) = (1+t, 2t+1)$

$\overrightarrow{r}(1) = (2,3)$

Thus $s = \int_{1}^{2} \frac{\sqrt{1+4t}}{t} \, dt = [\sqrt{5} \ln 2]$
7.3 Isometries

An isometry $f$ is a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$d((x_1, y_1), (x_2, y_2)) = d(f(x_1, y_1), f(x_2, y_2))$$

In Euclidean space, we define the following:

$T_{(a, b)} : \mathbb{R}^2 \to \mathbb{R}^2$ (translation)

$T_{(a, b)}(x, y) = (x + a, y + b)$.

Ex: Show that $T_{(a, b)}$ is an isometry

$T_{(a, b)}(x_1, y_1) = (x_1 + a, y_1 + b)$

$T_{(a, b)}(x_2, y_2) = (x_2 + a, y_2 + b)$

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$d(T_{(a, b)}(x_1, y_1), T_{(a, b)}(x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
Let \( R_\theta : \mathbb{R}^2 \to \mathbb{R}^2 \)

\[
R_\theta(x, y) = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix}
x \cos \theta - y \sin \theta \\
x \sin \theta + y \cos \theta
\end{bmatrix}
\]

\( R_\theta \) is an isometry.

a) Compute \( R_\pi(x, y) \)

b) Compute \( R_{\pi/4}(4, 0) \)

Ex Let \( S_\lambda : \mathbb{R}^2 \to \mathbb{R}^2 \)

\[
S_\lambda(x, y) = (\lambda x, \lambda y)
\]

a) Find the distance between \((2, 3)\) and \((1, 2)\) in PHP

b) Find the distance between \(S_2(2, 3)\) and \(S_2(4, 2)\) in PHP.
c) What do you conclude?
7.4 Inversion in the circle

Def: The Image of $P$ under inversion in the circle centered at $O$ with radius $r$ is the point $P'$ on the ray $OP$ such that

$$|OP'| = \frac{r^2}{|OP|}$$
Ex: Find the image of $P$ under the inversion in the circle $x^2 + y^2 = 1$, where

a) $P = (4, 0)$
b) $Q = (2, 3)$

Solution

a) $P' = P = (1, 0)$
b) \( |OQ'| = \sqrt{4 + q^2} = \sqrt{13} \) 

Equation of line \( OQ \) is 

\[ y = \frac{3}{2} x \]

Thus, \( Q' = (x, \frac{3}{2} x) \).

We need to find \( x \).

\[ |OQ'| = \sqrt{x^2 + \frac{9}{4} x^2} = \sqrt{\frac{4x^2 + 9x^2}{4}} \]

\[ = \sqrt{\frac{13x^2}{4}} = \frac{\sqrt{13} x}{2} = \frac{4}{\sqrt{13}} \]

\[ \sqrt{13} x = \frac{2}{\sqrt{13}} \Rightarrow x = \frac{2}{13} \]

Thus, \( Q' = \left( \frac{2}{13}, \frac{3}{13} \right) \).
Lemma 7.4.1  Let $l$ be a line which does not go through the origin $O$. The image of $l$ under inversion in the unit circle is a circle which goes through the origin $O$. 

![Diagram with points P, Q, O, and line l intersecting the circle at point Q']
Find the image of the vertical line \( x = a \) under inversion of the unit circle.

We obtain

\[
(x - \frac{1}{4})^2 + y^2 = \frac{1}{16}
\]
Ex.

Find the image of the region bounded between the lines \( x = 4, \quad x = 2 \)
under the inversion of the unit circle.
Lemma 7.4.2. Suppose \( \Gamma \) is a circle which does not go through the origin \( O \). Then the image of \( \Gamma \) under inversion in the unit circle is a circle.

Lemma 7.4.3. Inversions preserve angles.

Definition. The angles between 2 curves intersecting at \( P \) is the angle between the tangent lines passing through \( P \).
Let $\gamma$ be a $2 \times 2$ matrix such that
\[
\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad ad - bc \neq 0
\]

Define $T_\gamma(z) = \frac{az + b}{cz + d}$, where $z \in \mathbb{C}$.

Moreover, we accept the following facts:

- $T_\gamma(-\frac{1}{c}) = \infty$
- $T_\gamma(\infty) = \frac{a}{c}$ if $c \neq 0$
- $T_\gamma(\infty) = \infty$ if $c = 0$

**Ex** Put $\gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the identity matrix.

$T_\gamma(z) = z$. 

Ex. \( Y = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \), \( 4+3 \neq 0 \).

\[ T_Y (3) = \frac{23 - 1}{33 + 8} \]

**Theorem 76.1** Let \( Y_1, Y_2 \) be a invertible 2x2 matrices

\[ T_{Y_1 Y_2} (3) = T_{Y_1} (T_{Y_2} 3) \]

**Remark** As long as \( Y \) is invertible, then the map \( T_Y \) is one-to-one and invertible also, and if \( Y = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \),

\[ T^{-1}_Y = T_{Y^{-1}} \], where

\[ Y^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \]
Ex Let $\gamma = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$, $\delta = 1+i$

Compute $T_\gamma(1+i)$

$$T_\gamma(1+i) = \frac{1+i+3}{2(1+i)+1} = \frac{4+i}{2+2i+1}$$

$$= \frac{4+i}{3+2i}$$

$$= \frac{(4+i)(3-2i)}{13}$$

$$= \frac{12-8i+3i+2}{13}$$

$$= \frac{14-5i}{13}$$
Ex Let \( \gamma = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \).

Find \( T_{\gamma}^{-1}(\mathbf{z}) \) where \( \mathbf{z} = x + iy + c \).

\[
\gamma^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}
\]

\[
T_{\gamma}^{-1}(\mathbf{z}) = \frac{\frac{2}{3} x - \frac{1}{3} y}{-\frac{1}{3} y + \frac{2}{3} x}
\]

Lemma The horizontal translation by a

\[
T_{a}(x, y) = (x + a, y)
\]

can be thought of as a fractional linear transformation.

Identify \((x, y)\) with \(x + iy\)

\[
(x + a, y) = (x, y) + (a, 0)
\]

equivalent to \(x + iy + a\).
Let $\gamma = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$

$T_\gamma(z) = \frac{z + a}{z^2 + 1} = z + a$.

**Lemma 7.6.3** The map

$\Psi(x+iy) = \left( \frac{-x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right)$

which is the inversion in the unit circle followed by reflection through $x=0$ can be thought of as a fractional linear transformation.

**Proof**

$\Psi(z) = \Psi(x+iy)$

$= \frac{-x+iy}{x^2 + y^2}$

$= \frac{-(x-iy)}{(x+iy)(x-iy)}$

$= -\frac{1}{z}$
\( \psi(3) = \frac{0^3 - 1}{1^3 + 0} \). Thus

\( \psi \) can be thought of as

\[ Ty \text{ where } y = \begin{bmatrix} a & -1 \\ 1 & 0 \end{bmatrix}. \]
7.9 The cross ratio

**Definition** Let $a, b, c,$ and $d$ be 4 elements of $\mathbb{C} \setminus \{\infty\}$, at least 3 of which are distinct. We define the cross ratio of $a, b, c$, and $d$ to be

$$
(a, b; c, d) = \frac{a - c}{a - d} \frac{b - c}{b - d}
$$

**Remark**

$$
T_z = (z; a, b, c) = \frac{z - b}{z - c} \frac{a - b}{a - c}
$$

$$
T_a = \frac{a - b}{a - c} = 1
$$

$$
T_b = \frac{0}{a - b} = 0
$$
\[ T(c) = \frac{c-b}{c-c} = \infty \]

Ex. Find the fractional linear transformation which sends 1 to 4, \(-i\) to 0, \(-1\) to \(\infty\)

\[ T_z = (3+i, 4, -i, -1) \]

\[ = \left( \frac{3+i}{z+4} \right) / \left( \frac{1+i}{1+i} \right) \]

\[ = \left( \frac{3+i}{z+4} \right) \times \frac{2}{1+i} \]

\[ = \frac{2z+2i}{(3+i)(1+i)} \]
Transformations of the Euclidean Plane
Math 325

Let us consider the matrix

\[
M = \begin{bmatrix}
1 & 1 \\
-1 & 1 \\
\end{bmatrix}
\]

Also known as the quincunx matrix. We can think of this matrix as being a transformation of the Euclidean plane. Let

\[
\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2
\]

We compute the image of the point above under the transformation \( M \).

\[
\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ y - x \end{bmatrix}.
\]

**Theorem 1** Let \( S \) be the \( 1 \times 1 \) square with corners

\[
\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

The image of \( S \) under the transformation \( M \) is a \( \sqrt{2} \times \sqrt{2} \) square. Thus, \( M \) is not an isometry.

**Proof.** We would like to find the image of the square under the transformation induced by the matrix \( M \).

\[
\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},
\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},
\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.
\]

See below, in blue, the square \( S \) and its image under the transformation of the matrix \( M \).
Remark 2 Notice that the area of the image of the square above has for area 2 which is equal to the determinant of the matrix $M$.

Theorem 3 The image of the unit circle under the matrix $M$ is a circle centered at the origin with radius $\sqrt{2}$

Proof. Let $S$ be the unit circle. We may use a parametrization of the unit circle as follows.

$$S = \left\{ \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} : t \in [0,2\pi] \right\}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} = \begin{bmatrix} \cos t + \sin t \\ \sin t - \cos t \end{bmatrix}$$

Notice that

$$(\cos t + \sin t)^2 + (\sin t - \cos t)^2 = 2$$

Thus, the image of the unit circle, is a circle centered at the origin with radius $\sqrt{2}$. The unit circle is plotted below in blue, and its image is in red.

Let us consider the matrix

$$M = \begin{bmatrix} 2 & 0 \\ 1 & 1/2 \end{bmatrix}$$

Theorem 4 Let $S$ be the $1 \times 1$ square with corners

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

The image of $S$ under the transformation $M$ is (not a square) a parallelogram. Moreover the area of the parallelogram is equal to $\det M$.

Proof. We have

$$\begin{bmatrix} 2 & 0 \\ 1 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 1 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 1 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 1 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3/2 \end{bmatrix}$$
Thus, the image of $S$ has for corners

\[
\begin{bmatrix}
0 \\
0 \\
1 \\
\frac{1}{2} \\
\frac{3}{2}
\end{bmatrix}
\]

See below, in blue, the square $S$ and its image under the transformation of the matrix $M$.

![Image showing the square and its image under transformation]

and it is easy to see that the image of $S$ has for area $1$ which is also equal to the determinant of

\[
\begin{bmatrix}
2 & 0 \\
1 & \frac{1}{2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 \\
0 \\
1 \\
1
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 \\
0 \\
1 \\
1
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 \\
0 \\
1 \\
1
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 \\
0 \\
1 \\
1
\end{bmatrix}
\]

**Exercise 5** Let

Let $S$ be the $1 \times 1$ square with corners

\[
\begin{bmatrix}
0 \\
0 \\
1 \\
1
\end{bmatrix}
\]

Find the image of $S$ under the transformation $M$. Is $M$ an isometry?

**Exercise 6** Let

Let $S$ be the $1 \times 1$ square with corners

\[
\begin{bmatrix}
0 \\
0 \\
1 \\
1
\end{bmatrix}
\]

Find the image of $S$ under the transformation $M$. Is $M$ an isometry? Find the area of the image of the square $S$ under the transformation $M$. 

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