Double Integrals

1. Write a double integral \( \iint_{R} f(x, y) \, dA \) which gives the volume of the top half of a solid ball of radius 5. (You need to specify a function \( f(x, y) \) as well as a region \( R \).)

2. (a) If \( R \) is any region in the plane (\( \mathbb{R}^2 \)), what does the double integral \( \iint_{R} 1 \, dA \) represent? Why?

(b) Suppose the shape of a flat plate is described as a region \( R \) in the plane, and \( f(x, y) \) gives the density of the plate at the point \((x, y)\) in kilograms per square meter. What does the double integral \( \iint_{R} f(x, y) \, dA \) represent? Why?

3. If \( R \) is the rectangle \([1, 2] \times [3, 4]\), compute the double integral \( \iint_{R} 6x^2y \, dA \).
4. If $\mathcal{R}$ is the rectangle $[0, 1] \times [-1, 2]$, compute the double integral $\iint_{\mathcal{R}} 2ye^x \, dA$.

5. Find the volume of the solid that lies under $z = x^2 + y^2$ and above the square $0 \leq x \leq 2, -1 \leq y \leq 1$.

6. Find the volume of the solid enclosed by the surfaces $z = 4 - x^2 - y^2$, $z = x^2 + 2y^2 - 2$, $x = -1, x = 1$, $y = -1$, and $y = 1$. 

Double Integrals over General Regions

1. Let $\mathcal{R}$ be the region in the plane bounded by the lines $y = 0$, $x = 1$, and $y = 2x$. Evaluate the double integral $\iint_{\mathcal{R}} 2xy \, dA$.

2. Let $\mathcal{R}$ be the region bounded by $y = x^2$ and $y = 1$. Write the double integral $\iint_{\mathcal{R}} f(x, y) \, dA$ as an iterated integral in both possible orders.

3. For many regions, one order of integration will be simpler to deal with than the other. That is the case in this problem: use the shape of the region to decide which order of integration to use. Why is the other order more difficult?

Let $\mathcal{R}$ be the trapezoid with vertices $(0, 0)$, $(2, 0)$, $(1, 1)$, and $(0, 1)$. Write the double integral $\iint_{\mathcal{R}} f(x, y) \, dA$ as an iterated integral.
4. Sometimes, when converting a double integral to an iterated integral, we decide the order of integration based on the integrand, rather than the shape of the region — some integrands are easy to integrate with respect to one variable and much harder (or even impossible) to integrate with respect to the other. That is the case in this problem.

Evaluate the double integral \[
\int \int_{\mathcal{R}} \sqrt{y^3 + 1} \, dA
\]
where \(\mathcal{R}\) is the region in the first quadrant bounded by \(x = 0\), \(y = 1\), and \(y = \sqrt{x}\). (To decide the order of integration, first think about whether it’s easier to integrate the integrand with respect to \(x\) or with respect to \(y\).)

5. In each part, you are given an iterated integral. Sketch the region of integration, and then change the order of integration.

(a) \[
\int_{0}^{4} \int_{0}^{x} f(x, y) \, dy \, dx.
\]

(b) \[
\int_{0}^{4} \int_{0}^{\sqrt{y}} f(x, y) \, dx \, dy.
\]

(c) \[
\int_{0}^{1} \int_{\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y) \, dx \, dy.
\]
6. Let \( a \) be a constant between 0 and 4. Let \( \mathcal{R} \) be the region bounded by \( y = x^2 + a \) and \( y = 4 \). Write the double integral \( \iint_{\mathcal{R}} f(x, y) \, dA \) as an iterated integral in both possible orders.

7. Evaluate the iterated integral \( \int_0^1 \int_{-\sqrt{1-x^2}}^0 2x \cos \left( y - \frac{y^3}{3} \right) \, dy \, dx \).

More problems on the other side!
8. A flat plate is in the shape of the region in the first quadrant bounded by \( x = 0, y = 0, y = \ln x \) and \( y = 2 \). If the density of the plate at point \((x, y)\) is \( xe^y \) grams per cm\(^2\), find the mass of the plate. (Suppose the \( x\)- and \( y\)-axes are marked in cm.)

9. Let \( U \) be the solid above \( z = 0 \), below \( z = 4 - y^2 \), and between the surfaces \( x = \sin y - 1 \) and \( x = \sin y + 1 \). Find the volume of \( U \).
1. A flat plate is in the shape of the region \( \mathcal{R} \) in the first quadrant lying between the circles \( x^2 + y^2 = 1 \) and \( x^2 + y^2 = 4 \). The density of the plate at point \((x, y)\) is \( x + y \) kilograms per square meter (suppose the axes are marked in meters). Find the mass of the plate.

2. Find the area of the region \( \mathcal{R} \) lying between the curves \( r = 2 + \sin 3\theta \) and \( r = 4 - \cos 3\theta \). (You may leave your answer as an iterated integral in polar coordinates.)

3. In each part, rewrite the double integral as an iterated integral in polar coordinates. (Do not evaluate.)

   (a) \[ \int_{\mathcal{R}} \sqrt{1 - x^2 - y^2} \, dA \] where \( \mathcal{R} \) is the left half of the unit disk.
(b) \[ \iint_{R} x^2 \, dA \] where \( R \) is the right half of the ring \( 4 \leq x^2 + y^2 \leq 9 \).

4. Rewrite the iterated integral in Cartesian coordinates \( \int_{0}^{2} \int_{-\sqrt{1-y^2}}^{\sqrt{4-y^2}} xy \, dx \, dy \) as an iterated integral in polar coordinates. (Try to draw the region of integration.) You need not evaluate.

5. Find the volume of the solid enclosed by the \( xy \)-plane and the paraboloid \( z = 9 - x^2 - y^2 \). (You may leave your answer as an iterated integral in polar coordinates.)

6. The region inside the curve \( r = 2 + \sin 3\theta \) and outside the curve \( r = 3 - \sin 3\theta \) consists of three pieces. Find the area of one of these pieces. (You may leave your answer as an iterated integral in polar coordinates.)
When doing integrals in polar coordinates, you often need to integrate trigonometric functions. The **double-angle formulas** are very useful for this. (For instance, they are helpful for the integral in #2.)

The double-angle formulas are easily derived from the fact

\[ e^{it} = \cos t + i \sin t \]  

(1)

If \( \theta \) is any angle, then

\[ e^{i\theta} e^{i\theta} = e^{2i\theta}. \]

Using (1) with \( t = \theta \) on the left and \( t = 2\theta \) on the right, this becomes

\[ (\cos \theta + i \sin \theta)(\cos \theta + i \sin \theta) = \cos 2\theta + i \sin 2\theta \]

\[ \cos^2 \theta - \sin^2 \theta + 2i \sin \theta \cos \theta = \cos 2\theta + i \sin 2\theta \]

Equating the real parts of both sides, \[ \cos^2 \theta - \sin^2 \theta = \cos 2\theta. \] Equating the imaginary parts, \[ 2\sin \theta \cos \theta = \sin 2\theta. \]

The formula \[ \cos 2\theta = \cos^2 \theta - \sin^2 \theta \] also leads to useful identities for \( \cos^2 \theta \) and \( \sin^2 \theta \):

\[
\begin{align*}
\cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\
&= \cos^2 \theta - (1 - \cos^2 \theta) \\
&= 2 \cos^2 \theta - 1 \\
\cos^2 \theta &= \frac{1}{2}(1 + \cos 2\theta)
\end{align*}
\]

\[
\begin{align*}
\cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\
&= (1 - \sin^2 \theta) - \sin^2 \theta \\
&= 1 - 2 \sin^2 \theta \\
\sin^2 \theta &= \frac{1}{2}(1 - \cos 2\theta)
\end{align*}
\]

These two identities make it easy to integrate \( \sin^2 \theta \) and \( \cos^2 \theta \).

For the remaining problems, use polar coordinates or Cartesian coordinates, whichever seems easier.

7. Find the volume of the “ice cream cone” bounded by the single cone \( z = \sqrt{x^2 + y^2} \) and the paraboloid \( z = 3 - \frac{x^2}{4} - \frac{y^2}{4} \).
8. A flat plate is in the shape of the region $R$ defined by the inequalities $x^2 + y^2 \leq 4$, $0 \leq y \leq 1$, $x \leq 0$. The density of the plate at the point $(x, y)$ is $-xy$. Find the mass of the plate.

9. Find the area of the region which lies inside the circle $x^2 + (y-1)^2 = 1$ but outside the circle $x^2 + y^2 = 1$. 
1. Write a double integral \( \iint_{R} f(x, y) \, dA \) which gives the volume of the top half of a solid ball of radius 5. (You need to specify a function \( f(x, y) \) as well as a region \( R \).)

**Solution.** We know that \( \iint_{R} f(x, y) \) can be interpreted as the volume of the solid under \( z = f(x, y) \) over the region \( R \). So, we’d like to think of a function \( f(x, y) \) and a region \( R \) so that the solid under \( z = f(x, y) \) over \( R \) is the top half of a solid ball of radius 5.

Let’s look at the sphere of radius 5 centered at the origin. We know that it has equation \( x^2 + y^2 + z^2 = 25 \), so the top half of this sphere can be described by \( z = \sqrt{25 - x^2 - y^2} \). The solid under \( z = \sqrt{25 - x^2 - y^2} \) and over the region \( x^2 + y^2 \leq 25 \) is therefore half of a solid ball of radius 5. So, the double integral \( \iint_{R} \sqrt{25 - x^2 - y^2} \, dA \) where \( R \) is the disk \( x^2 + y^2 \leq 25 \) gives the volume of the top half of a solid ball of radius 5.

2. (a) If \( R \) is any region in the plane \((\mathbb{R}^2)\), what does the double integral \( \iint_{R} 1 \, dA \) represent? Why?

**Solution.** Remember that we are thinking of the double integral \( \iint_{R} f(x, y) \, dA \) as a limit of Riemann sums, obtained from the following process:

1. Slice the region \( R \) into small pieces.
2. In each piece, the value of \( f \) will be approximately constant, so multiply the value of \( f \) at any point by the area \( \Delta A \) of the piece.\(^{(1)} \)
3. Add up all of these products. (This is a Riemann sum.)
4. Take the limit of the Riemann sums as the area of the pieces tends to 0.

Now, if \( f \) is just the function \( f(x, y) = 1 \), then in Step 2, we end up simply multiplying 1 by the area of the piece, which gives us the area of the piece. So, in Step 3, when we add all of these products up, we are just adding up the area of all the small pieces, which gives the area of the whole region.

So, \( \iint_{R} 1 \, dA \) represents the area of the region \( R \).

(b) Suppose the shape of a flat plate is described as a region \( R \) in the plane, and \( f(x, y) \) gives the density of the plate at the point \((x, y)\) in kilograms per square meter. What does the double integral \( \iint_{R} f(x, y) \, dA \) represent? Why?

**Solution.** Following the process described in (a), in Step 2, we multiply the approximate density of each piece by the area of that piece, which gives the approximate mass of that piece. Adding those up gives the approximate mass of the entire plate, and taking the limit gives us the exact mass of the plate.

\(^{(1)}\) Actually, it’s also fine to just approximate the area of the piece.
3. If $R$ is the rectangle $[1,2] \times [3,4]$, compute the double integral $\int_R 6x^2y \, dA$.

**Solution.** We can rewrite this as an integrated integral in two ways: $\int_1^2 \left( \int_3^4 6x^2y \, dy \right) \, dx$ or $\int_3^4 \left( \int_1^2 6x^2y \, dx \right) \, dy$. These will give the same answer (that’s what Fubini’s Theorem says), so let’s just use the first. We need to first do the inner integral, which is $\int_3^4 6x^2y \, dy$. When we do this integral, we treat $x$ as a constant. So, this integral is equal to $3x^2y^2 \bigg|_{y=3}^{y=4} = 3x^2(16 - 9) = 21x^2$. So, our iterated integral becomes $\int_1^2 21x^2 \, dx = 7x^3 \bigg|_{x=1}^{x=2} = 49$.

4. If $R$ is the rectangle $[0,1] \times [-1,2]$, compute the double integral $\int_R 2ye^x \, dA$.

**Solution.** We can rewrite the double integral as an iterated integral in two ways: $\int_0^1 \int_{-1}^2 2ye^x \, dy \, dx$ or $\int_{-1}^2 \int_0^1 2ye^x \, dx \, dy$. Let’s use the first to compute.

$$
\int_0^1 \int_{-1}^2 2ye^x \, dy \, dx = \int_0^1 \left( y^2e^x \bigg|_{y=-1}^{y=2} \right) \, dx
= \int_0^1 3e^x \, dx
= 3e^x \bigg|_{x=0}^{x=1}
= 3e - 3
$$

5. Find the volume of the solid that lies under $z = x^2 + y^2$ and above the square $0 \leq x \leq 2$, $-1 \leq y \leq 1$.

**Solution.** We know that the volume of the solid lying under a surface $z = f(x,y)$ and above a region $\mathcal{R}$ in the plane is given by the double integral $\iint_{\mathcal{R}} f(x,y) \, dA$, so the volume we want in this problem is given by the double integral $\iint_{\mathcal{R}} (x^2 + y^2) \, dA$ where $\mathcal{R}$ is the square $[0,2] \times [-1,1]$. We know that this double integral is equal to the iterated integrals $\int_0^2 \int_{-1}^1 (x^2 + y^2) \, dy \, dx$ and $\int_{-1}^1 \int_0^2 (x^2 + y^2) \, dx \, dy$. 

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2
Let’s use the first iterated integral:

\[
\int_0^2 \int_{-1}^1 (x^2 + y^2) \, dy \, dx = \int_0^2 \left[ \left( x^2y + \frac{y^3}{3} \right) \bigg|_{y=-1}^{y=1} \right] \, dx
\]

\[
= \int_0^2 \left( 2x^2 + \frac{2}{3} \right) \, dx
\]

\[
= \frac{2x^3}{3} + \frac{2x}{3} \bigg|_{x=2}^{x=0}
\]

\[
= \frac{20}{3}
\]

6. Find the volume of the solid enclosed by the surfaces 
\[ z = 4 - x^2 - y^2, z = x^2 + 2y^2 - 2, x = -1, x = 1, \]
\[ y = -1, \text{ and } y = 1. \]

Solution. When we study triple integrals, we’ll see another way to do this problem.

First, let’s figure out what this solid looks like. The surface 
\[ z = 4 - x^2 - y^2 \]
is a paraboloid which opens downward, with its highest point at \((0, 0, 4)\). The surface 
\[ z = x^2 + 2y^2 - 2 \]
is a paraboloid which opens upward, with its lowest point at \((0, 0, -2)\). So, here is a picture of the solid:

Here, the top surface is \( z = 4 - x^2 - y^2 \), and the bottom is \( z = x^2 + 2y^2 - 2 \). To find the volume of the solid, let’s imagine approximating it using boxes:

Basically, what we are doing is chopping the rectangle \( R = [-1, 1] \times [-1, 1] \) into lots of small rectangles, each of area \( \Delta A \). Then we look at a particular box:
Its volume is the area $\Delta A$ multiplied by the height of the box. The height of the box is the difference between the $z$-value at the top (on the surface $z = 4 - x^2 - y^2$) and the bottom (on the surface $z = x^2 + 2y^2 - 2$). So, its volume is approximately $[(4 - x^2 - y^2) - (x^2 + 2y^2 - 2)]\Delta A = (6 - 2x^2 - 3y^2)\Delta A$.

If we add all of these up and take the limit as $\Delta A \to 0$, we get the double integral $\iint_R (6 - 2x^2 - 3y^2) \, dA$, which we compute by converting to an iterated integral:

$$\begin{align*}
\iint_R (6 - 2x^2 - 3y^2) \, dA &= \int_{-1}^{1} \int_{-1}^{1} (6 - 2x^2 - 3y^2) \, dy \, dx \\
&= \int_{-1}^{1} \left( 6y - 2x^2y - y^3 \bigg|_{y=-1}^{y=1} \right) \, dx \\
&= \int_{-1}^{1} (10 - 4x^2) \, dx \\
&= \left[ 10x - \frac{4}{3}x^3 \right]_{x=-1}^{x=1} \\
&= \frac{52}{3}
\end{align*}$$
Double Integrals over General Regions

1. Let $\mathcal{R}$ be the region in the plane bounded by the lines $y = 0$, $x = 1$, and $y = 2x$. Evaluate the double integral $\iint_{\mathcal{R}} 2xy \, dA$.

Solution. We can either slice the region $\mathcal{R}$ vertically or horizontally.\(^{(1)}\)

- Slicing vertically:

Slicing vertically amounts to slicing the interval $[0, 1]$ on the $x$-axis, so our outer integral will be $\int_0^1$ something $dx$. To figure out the inner integral, we look at a general slice. Remember that, on a single slice, $x$ is (roughly) constant, and we want to describe what $y$ does. The bottom of each slice is on the line $y = 0$, and the top is on the line $y = 2x$, so the inner integral has endpoints of integration $0$ and $2x$. Therefore, our iterated integral is

\[
\int_0^1 \left( \int_0^{2x} 2xy \, dy \right) dx = \int_0^1 \left( xy^2 \right|_{y=0}^{y=2x} \right) dx = \int_0^1 4x^3 \, dx = x^4 \bigg|_{x=1}^{x=0} = 1
\]

- Slicing horizontally:

Slicing horizontally amounts to slicing the interval $[0, 2]$ on the $y$-axis, so our outer integral will be $\int_0^2$ something $dy$. To figure out the inner integral, we look at a general slice. The left end of each slice is on the line $y = 2x$, and the right end is on the line $x = 1$. Since we are describing

\(^{(1)}\)Remember that this is a streamlined version of the real process. Really, to get a Riemann sum approximation, we chop the region $\mathcal{R}$ into lots of small rectangles, each of width $\Delta x$ and height $\Delta y$. The area of each piece is then $\Delta A = \Delta x \Delta y$. We have one product $“f(x, y)\Delta x\Delta y”$ per little rectangle, and we need to add these all up to get a Riemann sum. (See #2 of the worksheet “Double Integrals” for more details.) When converting to an iterated integral, we’re really deciding whether we want to add up in rows or columns first. If we add up in rows, we visualize adding up in a horizontal slice first and getting one sum per horizontal slice (then we add up all of those sums, one per slice). Similarly, if we add up in columns, we visualize adding up in a vertical slice first and then adding up all those sums, one per vertical slice. So, when we say “slice horizontally,” we really mean we’re going to add up in rows first.
a horizontal slice, we want to describe how $x$ varies, so $x$ goes from $\frac{y}{2}$ to 1. Thus, the iterated integral is $\int_0^1 \int_{y/2}^1 2xy \, dx \, dy$, which is of course also equal to 1.

2. Let $\mathcal{R}$ be the region bounded by $y = x^2$ and $y = 1$. Write the double integral $\iint_{\mathcal{R}} f(x,y) \, dA$ as an iterated integral in both possible orders.

**Solution.** Again, we think of slicing either vertically or horizontally.

• **Slicing vertically:**

  Slicing vertically amounts to slicing the interval $[-1, 1]$ on the $x$-axis, so the outer integral will be $\int_{-1}^1$ something $dx$. To write the inner integral, we want to describe what $y$ does within a single slice (thinking of $x$ as being constant). The bottom of each slice lies on $y = x^2$, and the top lies on $y = 1$, so the iterated integral is $\int_{-1}^1 \int_{x^2}^1 f(x,y) \, dy \, dx$.

• **Slicing horizontally:**

  Slicing horizontally amounts to slicing the interval $[0, 1]$ on the $y$-axis, so the outer integral will be $\int_0^1$ something $dy$. The left side of each slice lies on $y = x^2$, and the right side of each slice also lies on $y = x^2$. Remember, though, that we are trying to describe how $x$ varies in a slice (and we think of $y$ as being constant), so $x$ goes from the left half of $y = x^2$, where $x = -\sqrt{y}$, to the right half, where $x = \sqrt{y}$. Thus, the iterated integral is $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y) \, dx \, dy$.

3. Let $\mathcal{R}$ be the trapezoid with vertices $(0, 0), (2, 0), (1, 1)$, and $(0, 1)$. Write the double integral $\iint_{\mathcal{R}} f(x,y) \, dA$ as an iterated integral.

**Solution.** Let’s compare slicing vertically with slicing horizontally:

Notice that, if we slice vertically, there are two “types” of slices. The slices to the left of $x = 1$ go from $y = 0$ to $y = 1$, whereas the slices to the right go from $y = 0$ to the diagonal side of the trapezoid.
In contrast, if we slice horizontally, all of the slices have the same description: they go from \( x = 0 \) to the diagonal side. This seems simpler, so let’s go with this method. When we slice horizontally, we are slicing the interval \([0, 1]\) on the \( y \)-axis, so our outer integral will be \( \int_0^1 \) something \( dy \). Each slice goes from \( x = 0 \) to the diagonal side. The diagonal side is \( y = 2 - x \) (we know it’s a line containing the points \((2, 0)\) and \((1, 1)\)). We want to describe how \( x \) varies in each slice, so \( x \) goes from 0 to \( 2 - y \).

So, the iterated integral is
\[
\int_0^1 \int_0^{2-y} f(x, y) \, dx \, dy \tag{2}
\]

4. Evaluate the double integral \( \iint_R \sqrt{y^3 + 1} \, dA \) where \( R \) is the region in the first quadrant bounded by \( x = 0 \), \( y = 1 \), and \( y = \sqrt{x} \). (To decide the order of integration, first think about whether it’s easier to integrate the integrand with respect to \( x \) or with respect to \( y \).)

**Solution.** The integrand is much easier to integrate with respect to \( x \) than with respect to \( y \). Therefore, we should try to rewrite the double integral as an iterated integral where the inner integral is with respect to \( x \). This means our outer integral will be with respect to \( y \), which corresponds in our strategy to slicing the region horizontally.

This amounts to slicing the interval \([0, 1]\) on the \( y \)-axis, so the outer integral will be \( \int_0^1 \) something \( dy \). Each slice has its left end on \( x = 0 \) and its right end on \( y = \sqrt{x} \). We want to describe how \( x \) varies within a slice, so we rewrite \( y = \sqrt{x} \) as \( x = y^2 \). This gives the iterated integral
\[
\int_0^1 \int_0^{y^2} \sqrt{y^3 + 1} \, dx \, dy = \int_0^1 \left( x \sqrt{y^3 + 1} \right)_{x=0}^{x=y^2} \, dy = \int_0^1 y^2 \sqrt{y^3 + 1} \, dy
\]

We can evaluate this integral using substitution: if we let \( u = y^3 + 1 \), then \( du = 3y^2 \, dy \), and we can rewrite the integral as
\[
\int_1^2 \frac{1}{3} \sqrt{u} \, du = \frac{2}{9} \left. u^{3/2} \right|_{u=1}^{u=2} = \frac{2}{9} (2^{3/2} - 1)
\]

\( \tag{2} \) If you used the other order of integration, you should have a sum of iterated integrals \( \int_0^1 \int_0^1 f(x, y) \, dy \, dx + \int_0^2 \int_0^{2-x} f(x, y) \, dy \, dx \).
5. In each part, you are given an iterated integral. Sketch the region of integration, and then change the order of integration.

(a) \( \int_0^4 \int_0^x f(x,y) \, dy \, dx \).

**Solution.** Let’s just think of our strategy in reverse. The fact that the outer integral is \( \int_0^4 \) something \( dx \) tells us that we are slicing the interval \([0, 4]\) on the \( x \)-axis, so we are making vertical slices from \( x = 0 \) to \( x = 4 \). The inner integral tells us that the bottom of each slice is on \( y = 0 \), and the top of each slice is on \( y = x \). So, the region of integration (with vertical slices) looks like the picture on the left:

![Vertical Slices](image1.png)

To change the order of integration, we want to instead use horizontal slices (the picture on the right). Now, we are slicing the interval \([0, 4]\) on the \( y \)-axis, so the outer integral is \( \int_0^4 \) something \( dy \). Each slice has its left edge on \( y = x \) (or \( x = y \), since we really want to describe \( x \) in terms of \( y \)) and its right edge on \( x = 4 \), so we can rewrite the iterated integral as \( \int_0^4 \int_y^4 f(x,y) \, dx \, dy \).

(b) \( \int_0^4 \int_0^{\sqrt{y}} f(x,y) \, dx \, dy \).

**Solution.** The fact that the outer integral is \( \int_0^4 \) something \( dy \) tells us that we are slicing the interval \([0, 4]\) on the \( y \)-axis, so we are making horizontal slices from \( y = 0 \) to \( y = 4 \). The inner integral tells us that the left side of each slice is on \( x = 0 \) and the right side is on \( x = \sqrt{y} \) (or \( y = x^2 \)). So, the region of integration looks like this:

![Horizontal Slices](image2.png)

To change the order of integration, we use vertical slices. Now, we are slicing the interval \([0, 2]\) on the \( x \)-axis. The bottom of each slice is on \( y = x^2 \), and the top of each slice is on \( y = 4 \), so we can rewrite the integral as \( \int_0^2 \int_{x^2}^4 f(x,y) \, dy \, dx \).
\( (c) \int_0^1 \int_{\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y) \, dx \, dy. \)

**Solution.** The fact that the outer integral is \( \int_0^1 \) something \( dy \) tells us that we are slicing the interval \([0, 1]\) on the \( y \)-axis, so we are making horizontal slices from \( y = 0 \) to \( y = 1 \). The inner integral tells us that the left side of each slice is on \( x = -\sqrt{1-y^2} \) and the right side of each slice is on \( x = \sqrt{1-y^2} \). \( x = -\sqrt{1-y^2} \) describes the left half of the circle \( x^2 + y^2 = 1 \), and \( x = \sqrt{1-y^2} \) describes the right half, so the region of integration looks like this:

![Region of integration](image)

To change the order of integration, we use vertical slices. Now, we are slicing the interval \([-1, 1]\) on the \( x \)-axis, so the outer integral is \( \int_{-1}^1 \) something \( dx \). Each slice has its bottom edge on \( y = 0 \) and its top edge on the top half of the circle \( x^2 + y^2 = 1 \) (or \( y = \sqrt{1-x^2} \)), so we can rewrite the iterated integral as

\[
\int_{-1}^1 \int_0^{\sqrt{1-x^2}} f(x, y) \, dy \, dx.
\]

6. Let \( a \) be a constant between 0 and 4. Let \( \mathcal{R} \) be the region bounded by \( y = x^2 + a \) and \( y = 4 \). Write the double integral \( \iint_{\mathcal{R}} f(x, y) \, dA \) as an iterated integral in both possible orders.

**Solution.** The curves \( y = x^2 + a \) and \( y = 4 \) intersect where \( x^2 = 4 - a \), so \( x = \pm \sqrt{4-a} \). So, the region \( \mathcal{R} \) looks like this:

![Region of integration](image)

To write the double integral as an iterated integral, we think of slicing either vertically or horizontally.
• Slicing vertically:

Slicing vertically corresponds to slicing the interval \([-\sqrt{4-a}, \sqrt{4-a}]\) on the x-axis, so the outer integral will be \(\int_{-\sqrt{4-a}}^{\sqrt{4-a}} \) something \(dx\). Each slice has its bottom edge on \(y = x^2 + a\) and its top edge on \(y = 4\), so the iterated integral is \(\int_{-\sqrt{4-a}}^{\sqrt{4-a}} \int_{x^2+a}^{4} f(x, y) \, dy \, dx\). Remember that \(a\) is a constant, so it’s fine to have it in the outer integral.

• Slicing horizontally:

Slicing horizontally corresponds to slicing the interval \([a, 4]\) on the y-axis, so the outer integral will be \(\int_{a}^{4} \) something \(dy\). Each slice has its left edge on \(y = x^2 + a\) (so \(x = -\sqrt{y-a}\)) and its right edge on \(y = x^2 + a\) (so \(x = \sqrt{y-a}\)). Thus, the iterated integral is \(\int_{a}^{4} \int_{-\sqrt{y-a}}^{\sqrt{y-a}} f(x, y) \, dx \, dy\).

7. Evaluate the iterated integral \(\int_{0}^{1} \int_{0}^{-\sqrt{1-x^2}} 2x \cos \left( y - \frac{y^3}{3} \right) \, dy \, dx\).

Solution. We don’t know how to integrate the integrand with respect to \(y\), but we can integrate it with respect to \(x\). This suggests that we should change the order of integration, as in #5. First, let’s figure out what the region looks like. The fact that the outer integral is \(\int_{0}^{1} \) something \(dx\) tells us that we are slicing the interval \([0, 1]\) on the x-axis, so we are making vertical slices from \(x = 0\) to \(x = 1\). The inner integral tells us that the bottom of each slice is on \(y = -\sqrt{1-x^2}\) (the bottom half of the circle \(x^2 + y^2 = 1\)) and the top of each slice is on \(y = 0\). So, the region of integration looks like this:
To change the order of integration, we switch to using horizontal slices. Now, we are slicing the interval 
$[-1, 0]$ on the $y$-axis, so our outer integral will be $\int_{-1}^{0}$ something $dy$. Each slice has its left edge on 
$x = 0$ and its right edge on the right half of the circle $x^2 + y^2 = 1$ (so $x = \sqrt{1 - y^2}$). Therefore, we 
can rewrite the given integral as

$$
\int_{-1}^{0} \int_{0}^{\sqrt{1-y^2}} 2x \cos\left(y - \frac{y^3}{3}\right) \, dx \, dy = \int_{-1}^{0} \left[ x^2 \cos\left(y - \frac{y^3}{3}\right) \right]_{0}^{\sqrt{1-y^2}} \, dy \\
= \int_{-1}^{0} (1 - y^2) \cos\left(y - \frac{y^3}{3}\right) \, dy
$$

We can use substitution to evaluate this integral: let $u = y - \frac{y^3}{3}$; then, $du = (1 - y^2)dy$, so the integral 
becomes

$$
\int_{-2/3}^{0} \cos u \, du = \sin u \bigg|_{u=0}^{u=-2/3} \\
= -\sin \left(\frac{-2}{3}\right)
$$

8. A flat plate is in the shape of the region in the first quadrant bounded by $x = 0$, $y = 0$, $y = \ln x$ and $y = 2$. If the density of the plate at point $(x, y)$ is $xe^y$ grams per cm$^2$, find the mass of the plate. 
(Suppose the $x$- and $y$-axes are marked in cm.)

**Solution.** As we learned in #2(b) of the worksheet “Double Integrals”, we can find the mass of the plate by taking the double integral of the density, where the region of integration is the plate. In this 
case, the integrand $xe^y$ is easy to integrate with respect to $x$ and with respect to $y$, so we will pick an 
order of integration based on the shape of the region. We can either slice horizontally or vertically:

As in #3, this region is simpler to describe using horizontal slices: with vertical slices, there are two 
“types” of slices, but with horizontal slices, there is only one.

If we use horizontal slices, we are slicing the interval $[0, 2]$ on the $y$-axis. Each slice goes from $x = 0$ to 
y = $\ln x$ (or $x = e^y$), so the iterated integral is

$$
\int_{0}^{2} \int_{0}^{e^y} xe^y \, dx \, dy = \int_{0}^{2} \left( \frac{1}{2} x^2 e^y \bigg|_{x=0}^{x=e^y} \right) \, dy \\
= \int_{0}^{2} \frac{1}{2} e^{3y} \, dy \\
= \frac{1}{6} e^{3y} \bigg|_{y=0}^{y=2} \\
= \frac{1}{6} (e^6 - 1)
$$
9. Let $U$ be the solid above $z = 0$, below $z = 4 - y^2$, and between the surfaces $x = \sin y - 1$ and $x = \sin y + 1$. Find the volume of $U$.

**Solution.** The picture on the left shows the four surfaces $z = 0$, $z = 4 - y^2$, $x = \sin y - 1$, and $x = \sin y + 1$. The picture on the right shows just the solid $U$.

This solid can be described as the solid under $z = 4 - y^2$ over the region $R$, where $R$ is where the solid meets the $xy$-plane. So, its volume will just be $\int \int_R (4 - y^2) \, dA$.

To calculate this double integral, we need to describe $R$ and convert the double integral to an iterated integral. The surface $z = 4 - y^2$ intersects the $xy$-plane $z = 0$ where $4 - y^2 = 0$, or $y = \pm 2$, so $y = 2$ and $y = -2$ are 2 boundaries of the region $R$. The other two are $x = \sin y - 1$ and $x = \sin y + 1$. So, $R$ looks like this:

It’s easier to slice this region horizontally:

This amounts to slicing the interval $[-2, 2]$ on the $y$-axis, so the outer integral will be $\int_{-2}^{2}$ something $dy$. The left side of each slice is on $x = \sin y - 1$, and the right side is on $x = \sin y + 1$, so we can rewrite
the double integral as an iterated integral

\[
\int_{-2}^{\sin y+1} \int_{\sin y-1}^{\sin y+1} (4 - y^2) \, dx \, dy = \int_{-2}^{2} \left[ x(4 - y^2) \right]_{x=\sin y-1}^{x=\sin y+1} \, dy
\]

\[
= \int_{-2}^{2} 2(4 - y^2) \, dy
\]

\[
= 8y - \frac{2y^3}{3} \bigg|_{y=-2}^{y=2}
\]

\[
= \frac{64}{3}
\]
Double Integrals in Polar Coordinates

1. A flat plate is in the shape of the region \( R \) in the first quadrant lying between the circles \( x^2 + y^2 = 1 \) and \( x^2 + y^2 = 4 \). The density of the plate at point \((x, y)\) is \( x + y \) kilograms per square meter (suppose the axes are marked in meters). Find the mass of the plate.

Solution. As we saw in #2(b) of the worksheet “Double Integrals”, the mass is the double integral of density. That is, the mass is \( \iint_{R} (x + y) \, dA \).

To compute double integrals, we always convert them to iterated integrals. In this case, we’ll use a double integral in polar coordinates. The region \( R \) is the polar rectangle \( 0 \leq \theta \leq \frac{\pi}{2} \), \( 1 \leq r \leq 2 \), so we can rewrite the double integral as an iterated integral in polar coordinates:

\[
\iint_{R} (x + y) \, dA = \int_{0}^{\pi/2} \int_{1}^{2} (r \cos \theta + r \sin \theta)(r \, dr \, d\theta)
\]

\[
= \int_{0}^{\pi/2} \int_{1}^{2} r^2 (\cos \theta + \sin \theta) \, dr \, d\theta
\]

\[
= \int_{0}^{\pi/2} \left( \int_{1}^{2} \frac{1}{3} r^3 (\cos \theta + \sin \theta) \, dr \right) \, d\theta
\]

\[
= \int_{0}^{\pi/2} \frac{7}{3} (\cos \theta + \sin \theta) \, d\theta
\]

\[
= \frac{7}{3} (\sin \theta - \cos \theta) \bigg|_{\theta=0}^{\theta=\pi/2}
\]

\[
= \frac{14}{3}
\]

2. Find the area of the region \( R \) lying between the curves \( r = 2 + \sin 3\theta \) and \( r = 4 - \cos 3\theta \). (You may leave your answer as an iterated integral in polar coordinates.)

Solution. As we saw in #2(a) of the worksheet “Double Integrals”, the area of the region \( R \) is equal to the double integral \( \iint_{R} 1 \, dA \). To compute the value of this double integral, we will convert it to an iterated integral.

This region is not a polar rectangle, so we’ll think about slicing. Let’s make slices where \( \theta \) is constant:
Our slices go all the way around the origin, so the outer integral will have $\theta$ going from 0 to $2\pi$. Along each slice, $r$ goes from the inner curve ($r = 2 + \sin 3\theta$) to the outer curve ($r = 4 - \cos 3\theta$). So, the iterated integral is

$$
\int_0^{2\pi} \int_{2+\sin 3\theta}^{4-\cos 3\theta} 1 \cdot r \, dr \, d\theta = \int_0^{2\pi} \left( \frac{1}{2} r^2 \bigg|_{r=2+\sin 3\theta}^{r=4-\cos 3\theta} \right) \, d\theta
$$

$$
= \frac{1}{2} \int_0^{2\pi} \left[ (4 - \cos 3\theta)^2 - (2 + \sin 3\theta)^2 \right] \, d\theta
$$

$$
= \frac{1}{2} \int_0^{2\pi} \left( 16 - 8 \cos 3\theta + \cos^2 3\theta - 4 - 4 \sin 3\theta - \sin^2 3\theta \right) \, d\theta
$$

$$
= \frac{1}{2} \int_0^{2\pi} \left( 12 - 8 \cos 3\theta + \cos 6\theta - 4 \sin 3\theta \right) \, d\theta
$$

by the double angle identity $\cos 2t = \cos^2 t - \sin^2 t$

$$
= \frac{1}{2} \left( 12\theta - \frac{8}{3} \sin 3\theta + \frac{1}{6} \sin 6\theta + \frac{4}{3} \cos 3\theta \right) \bigg|_{\theta=0}^{\theta=2\pi}
$$

$$
= 12\pi
$$

3. In each part, rewrite the double integral as an iterated integral in polar coordinates. (Do not evaluate.)

(a) $\int\int_{\mathcal{R}} \sqrt{1-x^2-y^2} \, dA$ where $\mathcal{R}$ is the left half of the unit disk.

Solution. The region $\mathcal{R}$ is the polar rectangle $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$, $0 \leq r \leq 1$. In polar coordinates, the integrand is $\sqrt{1-r^2}$. So, we can rewrite the double integral as an iterated integral

$$
\int_{\pi/2}^{3\pi/2} \int_0^1 \sqrt{1-r^2} \cdot r \, dr \, d\theta
$$
(b) \( \iint_{\mathcal{R}} x^2 \, dA \) where \( \mathcal{R} \) is the right half of the ring \( 4 \leq x^2 + y^2 \leq 9 \).

\[ \int \int_{\mathcal{R}} x^2 \, dA \]

**Solution.** The region \( \mathcal{R} \) is the polar rectangle \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \ 2 \leq r \leq 3 \). In polar coordinates, the integrand is \((r \cos \theta)^2\). So, we can rewrite the double integral as an iterated integral

\[ \int_{-\pi/2}^{\pi/2} \int_{2}^{3} r^2 \cos^2 \theta \cdot r \, dr \, d\theta. \]

4. Rewrite the iterated integral in Cartesian coordinates \( \int_{0}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} xy \, dx \, dy \) as an iterated integral in polar coordinates. (Try to draw the region of integration.) You need not evaluate.

**Solution.** Let’s first write the integrand in polar coordinates. Since \( x = r \cos \theta \) and \( y = r \sin \theta \), the integrand can be written as \( r^2 \sin \theta \cos \theta \).

Next, let’s figure out the region of integration. Since the outer integral is \( \int_{0}^{2} \) something \( dy \), we are slicing the interval \([0, 2]\) on the \( y \)-axis, so we are making horizontal slices from \( y = 0 \) to \( y = 2 \). The inner integral tells us that the left side of each slice is on \( x = -\sqrt{4-y^2} \) and the right side of each slice is on \( x = \sqrt{4-y^2} \). \( x = -\sqrt{4-y^2} \) is the left half of the circle \( x^2 + y^2 = 4 \), and \( x = \sqrt{4-y^2} \) is the right half of the circle \( x^2 + y^2 = 4 \), so our region of integration (with horizontal slices) looks like this:

This region is the polar rectangle \( 0 \leq \theta \leq \pi, \ 0 \leq r \leq 2 \). So, the integral in polar coordinates is

\[ \int_{0}^{\pi} \int_{0}^{2} r^2 \sin \theta \cos \theta \cdot r \, dr \, d\theta. \]

5. Find the volume of the solid enclosed by the \( xy \)-plane and the paraboloid \( z = 9 - x^2 - y^2 \). (You may leave your answer as an iterated integral in polar coordinates.)

\( \text{(1)} \) Normally, we want \( \theta \) to be between 0 and \( 2\pi \). However, if it’s more convenient for a polar integral, we relax this restriction.
Solution. Let’s break this down into two steps:

1. First, we’ll write a double integral expressing the volume.

2. Then, we’ll convert the double integral to an iterated integral.

Notice that the solid can be described as the solid under \( z = 9 - x^2 - y^2 \) over the region \( R \), where \( R \) is where the solid meets the \( xy \)-plane. So, its volume will be \( \iiint_{R} (9 - x^2 - y^2) \, dA \). Let’s describe \( R \) in more detail. The surface \( z = 9 - x^2 - y^2 \) intersects the \( xy \)-plane \( z = 0 \) where \( x^2 + y^2 = 9 \), so the region \( R \) is the disk \( x^2 + y^2 \leq 9 \).

Now, we’ll convert this double integral to an iterated integral. The region \( R \) is the polar rectangle \( 0 \leq \theta \leq 2\pi, 0 \leq r \leq 3 \), so we can rewrite the double integral as

\[
\iint_{R} (9 - x^2 - y^2) \, dA = \int_{0}^{2\pi} \int_{0}^{3} (9 - r^2) r \, dr \, d\theta
\]

\[
= \int_{0}^{2\pi} \int_{0}^{3} (9r - r^3) \, dr \, d\theta
\]

\[
= \int_{0}^{2\pi} \int_{0}^{3} \left( \frac{9r^2}{2} - \frac{r^4}{4} \bigg|_{r=3} \right) \, d\theta
\]

\[
= \int_{0}^{2\pi} \frac{81}{4} \, d\theta
\]

\[
= \frac{81\pi}{4} \bigg|_{\theta=2\pi}
\]

\[
= \frac{81\pi}{2}
\]

6. The region inside the curve \( r = 2 + \sin 3\theta \) and outside the curve \( r = 3 - \sin 3\theta \) consists of three pieces. Find the area of one of these pieces. (You may leave your answer as an iterated integral in polar coordinates.)

Solution. Since we are finding area, our integral will be \( \iint_{R} 1 \, dA \), where \( R \) is the region of integration. As always, to evaluate the double integral, we need to rewrite it as an iterated integral (this time, in polar coordinates).

Let’s make slices where \( \theta = \text{constant} \). (2)

(2) When we’re dealing with regions that aren’t polar rectangles, it’s almost always easier to slice where \( \theta = \text{constant} \).
We are slicing from the $\theta$ of the red point to the $\theta$ of the blue point. Let’s find these. The red point and blue point are points where the curves $r = 2 + \sin 3\theta$ and $r = 3 - \sin 3\theta$ intersect, so let’s solve $2 + \sin 3\theta = 3 - \sin 3\theta$. This happens when $\sin 3\theta = \frac{1}{2}$, or $3\theta = \frac{\pi}{6}, \frac{5\pi}{6}$. So, the red point has $\theta = \frac{\pi}{18}$, the blue point has $\theta = \frac{5\pi}{18}$, and our outer integral will have $\theta$ going from $\frac{\pi}{18}$ to $\frac{5\pi}{18}$.

Along a slice, $r$ goes from the inner curve ($r = 3 - \sin 3\theta$) to the outer curve ($r = 2 + \sin 3\theta$), so we can rewrite our double integral as

$$\int_{\pi/18}^{5\pi/18} \int_{3-\sin 3\theta}^{2+\sin 3\theta} 1 \cdot r \, dr \, d\theta = \int_{\pi/18}^{5\pi/18} \left( \frac{1}{2} r^2 \bigg|_{r=3-\sin 3\theta}^{r=2+\sin 3\theta} \right) \, d\theta$$

$$= \frac{1}{2} \int_{\pi/18}^{5\pi/18} \left[ (2 + \sin 3\theta)^2 - (3 - \sin 3\theta)^2 \right] \, d\theta$$

$$= \frac{1}{2} \int_{\pi/18}^{5\pi/18} (-5 + 10 \sin 3\theta) \, d\theta$$

$$= \frac{1}{2} \left( -5\theta - \frac{10}{3} \cos 3\theta \right) \bigg|_{\theta=\pi/18}^{\theta=5\pi/18}$$

$$= \frac{5 \sqrt{3} - 5\pi}{9}$$

7. Find the volume of the “ice cream cone” bounded by the single cone $z = \sqrt{x^2 + y^2}$ and the paraboloid $z = 3 - \frac{x^2}{4} - \frac{y^2}{4}$.

Solution. Let $\mathcal{R}$ be the projection of the solid onto the $xy$-plane; that is, let $\mathcal{R}$ be the region we see if we look down on the solid from above. This will be a disk, so let’s do the integral in polar coordinates.
First, we’ll rewrite everything in terms of polar coordinates. The cone \( z = \sqrt{x^2 + y^2} \) can be rewritten as \( z = r \), and the paraboloid \( z = 3 - \frac{x^2}{4} - \frac{y^2}{4} \) can be rewritten as \( z = 3 - \frac{r^2}{4} \).

To find the disk \( R \), notice that, if we look at the solid from above, the disk we see is the size of the circle where the two surfaces intersect. The surfaces intersect where \( r = 3 - \frac{r^2}{4} \); this can be rewritten as \( r^2 + 4r - 12 = 0 \), or \((r + 6)(r - 2) = 0\). Since \( r \geq 0 \), the intersection is \( r = 2 \). So, the region \( R \) is a disk centered at the origin with radius 2. This is a polar rectangle with \( 0 \leq r \leq 2, \ 0 \leq \theta < 2\pi \).

One way to find the volume of the solid is to find the volume under the paraboloid over \( R \), find the volume under the cone over \( R \), and subtract the latter from the former. That is:

\[
\text{volume under } z = 3 - \frac{r^2}{4} \quad \text{ over } R \\
\text{minus} \\
\text{volume under } z = r \quad \text{ over } R \\
\text{equals} \\
\text{volume we want}
\]

So, the iterated integral in polar coordinates is

\[
\int_0^{2\pi} \int_0^2 \left( 3 - \frac{r^2}{4} \right) r \, dr \, d\theta - \int_0^{2\pi} \int_0^2 r \cdot r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 \left( 3 - \frac{r^2}{4} - r \right) r \, dr \, d\theta \\
= \int_0^{2\pi} \int_0^2 \left( 3r - \frac{r^3}{4} - r^2 \right) \, dr \, d\theta \\
= \int_0^{2\pi} \left( \frac{3r^2}{2} - \frac{r^4}{16} - \frac{r^3}{3} \right)_{r=0}^{r=2} \, d\theta \\
= \int_0^{2\pi} \frac{7}{3} \, d\theta \\
= \frac{14\pi}{3}
\]

Notice that we end up simply integrating the difference between \( 3 - \frac{r^2}{4} \) and \( r \); this is really the height of the solid above the point \((r, \theta)\). For an explanation of why this works in terms of Riemann sums, see #6 of the worksheet “Double Integrals”.

8. A flat plate is in the shape of the region \( R \) defined by the inequalities \( x^2 + y^2 \leq 4, \ 0 \leq y \leq 1, \ x \leq 0 \). The density of the plate at the point \((x, y)\) is \(-xy\). Find the mass of the plate.

**Solution.** As we saw in #2(b) of the worksheet “Double Integrals”, the mass is the double integral of density. That is, the mass is \( \iint_R -xy \, dA \).

Here is a picture of the region:

\( ^{(3)} \)This is similar to what you were asked to do in the homework problem §12.3, #30.
There are four ways we could slice: two in Cartesian (vertically or horizontally) and two in polar (where \( \theta \) is constant or where \( r \) is constant). Here are pictures of all four:

vertical \((x = \text{constant})\)  
horizontal \((y = \text{constant})\)  
\(\theta = \text{constant}\)  
\(r = \text{constant}\)

When slicing vertically, along \( \theta = \text{constant} \), or along \( r = \text{constant} \), there are multiple “types” of slices. However, if we slice horizontally, there is only one “type” of slice. This suggests that we should go with horizontal slices. Slicing horizontally amounts to slicing the interval \([0, 1]\) on the \( y \)-axis, so the outer integral will be \( \int_{0}^{1} \) something \( dy \). Each slice has its left end on the left edge of the circle \( x^2 + y^2 = 4 \) (so where \( x = -\sqrt{4 - y^2} \)) and its right end on \( x = 0 \), so we can rewrite the double integral as

\[
\int_{0}^{1} \int_{-\sqrt{1-y^2}}^{0} -xy \, dx \, dy = \int_{0}^{1} \left( -\frac{1}{2}x^2y \bigg|_{x=-\sqrt{4-y^2}}^{x=0} \right) \, dy
\]

\[
= \int_{0}^{1} \frac{1}{2} (4 - y^2)y \, dy
\]

\[
= \frac{1}{2} \int_{0}^{1} (4y - y^3) \, dy
\]

\[
= \frac{1}{2} \left( 2y^2 - \frac{y^4}{4} \right) \bigg|_{y=1}^{y=0}
\]

\[
= \frac{7}{8}
\]

9. Find the area of the region which lies inside the circle \( x^2 + (y-1)^2 = 1 \) but outside the circle \( x^2 + y^2 = 1 \).

**Solution.** Here is a picture of the region, which we’ll call \( R \):
There are four ways we could slice: two in Cartesian (vertically or horizontally) and two in polar (where \( \theta \) is constant or where \( r \) is constant). Here are pictures of all four:

vertical \((x = \text{constant})\) \hspace{1cm} \text{horizontal} \((y = \text{constant})\) \hspace{1cm} \theta = \text{constant} \hspace{1cm} r = \text{constant}

When slicing vertically or horizontally, we can see that there are multiple “types” of slices. When slicing where \( \theta = \text{constant} \) or \( r = \text{constant} \), there is only one type of slice. So, let’s do this in polar coordinates.

First, let’s write the equations of the two circles in polar coordinates. The circle \( x^2 + y^2 = 1 \) is just \( r = 1 \). The circle \( x^2 + (y - 1)^2 = 1 \) is more complicated:

\[
\begin{align*}
x^2 + (y - 1)^2 &= 1 \\
(r \cos \theta)^2 + (r \sin \theta - 1)^2 &= 1 \\
r^2 \cos^2 \theta + r^2 \sin^2 \theta - 2r \sin \theta + 1 &= r \\
r^2(\cos^2 \theta + \sin^2 \theta) - 2r \sin \theta &= 0 \\
r^2 &= 2r \sin \theta \\
r &= 2 \sin \theta
\end{align*}
\]

(In the last step, we’ve divided both sides by \( r \); this is fine since \( r > 0 \) on the circle.\(^4\))

We’ll use the third picture, where we slice along \( \theta = \text{constant} \). (We can use the fourth as well, but we’re more used to doing polar integrals by slicing where \( \theta = \text{constant} \).) Here’s a picture with more detail.

\(^4\)Actually, \( r = 0 \) at the very bottom of the circle, but as it’s just one point, it doesn’t really matter.
We are slicing from the \( \theta \) of the red point to the \( \theta \) of the blue point. Let’s find these values. The red point and blue point are points where the curves \( r = 1 \) and \( r = 2 \sin \theta \) intersect, so let’s solve \( 1 = 2 \sin \theta \). This happens when \( \theta = \frac{\pi}{6} \) and \( \theta = \frac{5\pi}{6} \). So, the red point has \( \theta = \frac{\pi}{6} \), the blue point has \( \theta = \frac{5\pi}{6} \), and our outer integral will have \( \theta \) going from \( \frac{\pi}{6} \) to \( \frac{5\pi}{6} \).

Along each slice, \( r \) goes from the lower circle \( (r = 1) \) to the upper circle \( (r = 2 \sin \theta) \), so the inner integral will have \( r \) going from 1 to \( 2 \sin \theta \). So, we can rewrite our double integral as

\[
\int_{\pi/6}^{5\pi/6} \int_{1}^{2 \sin \theta} 1 \cdot r \ dr \ d\theta = \int_{\pi/6}^{5\pi/6} \left( \frac{1}{2} r^2 \right)_{r=1}^{r=2 \sin \theta} \ d\theta \\
= \int_{\pi/6}^{5\pi/6} \left( 2 \sin^2 \theta - \frac{1}{2} \right) \ d\theta \\
= \int_{\pi/6}^{5\pi/6} \left( \frac{1}{2} - \cos 2\theta \right) \ d\theta \\
\text{by the identity } \sin^2 \theta = \frac{1}{2} (1 - \cos 2\theta) \\
= \frac{\theta}{2} - \frac{1}{2} \sin 2\theta \bigg|^{\theta=5\pi/6}_{\theta=\pi/6} \\
= \frac{\pi}{3} + \frac{\sqrt{3}}{2}.
\]